52nd International Mathematical Olympiad

12 – 24 July 2011
Amsterdam
The Netherlands

Problem Shortlist with Solutions

IMO2011 Amsterdam

International Mathematical Olympiad Amsterdam 2011

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IMPORTANT

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The problem selection committee

Bart de Smit (chairman), Ilya Bogdanov, Johan Bosman, Andries Brouwer, Gabriele Dalla Torre, Géza Kós, Hendrik Lenstra, Charles Leytem, Ronald van Luijk, Christian Reiher, Eckard Specht, Hans Sterk, Lenny Taelman

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The committee gratefully acknowledges the receipt of 142 problem proposals by the following 46 countries:

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Algebra

A1

A1

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1 + a_2 + a_3 + a_4$, let p_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

A2

A2

Determine all sequences $(x_1, x_2, \dots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$

A3

A3

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x + y)) = f(x) + (2x + y)g(y)$$

for all real numbers x and y.

A4

A4

Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n + 1) - g(n + 1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\dots f}_{k}(n)\dots))$.

A5

A5

Prove that for every positive integer n, the set $\{2, 3, 4, \dots, 3n + 1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

A6

A6

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

A7

A7

Let a, b, and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

Combinatorics

C1

C1

Let $n > 0$ be an integer. We are given a balance and n weights of weight 2^0, $2^1, \ldots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

C2

C2

Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

C3

C3

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

C4

C4

Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

C5

C5

Let m be a positive integer and consider a checkerboard consisting of m by m unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn 90° clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard or prove that such a moment does not necessarily exist.

C6

C6

Let n be a positive integer and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to *appear* in W if there exist indices $k \leq \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

C7

C7

On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

Geometry

G1

G1

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to AB at B' and to AC at C'. Suppose also that the circumcenter O of the triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

G2

G2

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

G3

G3

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC, and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB. Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

G4

G4

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.

G5

G5

Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD, and KP are either parallel or concurrent.

G6

G6

Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

G7

G7

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

G8

G8

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

Number Theory

N1

N1

For any integer $d > 0$, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so for example we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f(2^k)$ divides $f(2^{k+1})$.

N2

N2

Consider a polynomial $P(x) = (x + d_1)(x + d_2) \cdot \ldots \cdot (x + d_9)$, where $d_1, d_2, \ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

N3

N3

Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

N4

N4

For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n such that all the differences

$$t(n+a) - t(n), \quad t(n+a+1) - t(n+1), \quad \ldots, \quad t(n+2a-1) - t(n+a-1)$$

are divisible by 4.

N5

N5

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

N6

N6

Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)} - 1$ divides $3^{P(n)} - 1$. Prove that $Q(x)$ is a constant polynomial.

N7

N7

Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.

N8

N8

Let k be a positive integer and set $n = 2^k + 1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1, \ldots, a_{n-1}$ of the numbers $1, 2, \ldots, n-1$ and a sequence of integers $g_1, g_2, \ldots, g_{n-1}$ such that n divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$, where we set $a_n = a_1$.

A1

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1+a_2+a_3+a_4$, let p_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

Answer. The sets A for which p_A is maximal are the sets the form $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer. For all these sets p_A is 4.

Solution. Firstly, we will prove that the maximum value of p_A is at most 4. Without loss of generality, we may assume that $a_1 < a_2 < a_3 < a_4$. We observe that for each pair of indices (i, j) with $1 \leq i < j \leq 4$, the sum $a_i + a_j$ divides s_A if and only if $a_i + a_j$ divides $s_A - (a_i + a_j) = a_k + a_l$, where k and l are the other two indices. Since there are 6 distinct pairs, we have to prove that at least two of them do not satisfy the previous condition. We claim that two such pairs are (a_2, a_4) and (a_3, a_4). Indeed, note that $a_2 + a_4 > a_1 + a_3$ and $a_3 + a_4 > a_1 + a_2$. Hence $a_2 + a_4$ and $a_3 + a_4$ do not divide s_A. This proves $p_A \leq 4$.

Now suppose $p_A = 4$. By the previous argument we have

$$\begin{array}{ccc} a_1 + a_4 \mid a_2 + a_3 & \text{and} & a_2 + a_3 \mid a_1 + a_4, \\ a_1 + a_2 \mid a_3 + a_4 & \text{and} & a_3 + a_4 \nmid a_1 + a_2, \\ a_1 + a_3 \mid a_2 + a_4 & \text{and} & a_2 + a_4 \nmid a_1 + a_3. \end{array}$$

Hence, there exist positive integers m and n with $m > n \geq 2$ such that

$$\begin{cases} a_1 + a_4 = a_2 + a_3 \\ m(a_1 + a_2) = a_3 + a_4 \\ n(a_1 + a_3) = a_2 + a_4. \end{cases}$$

Adding up the first equation and the third one, we get $n(a_1 + a_3) = 2a_2 + a_3 - a_1$. If $n \geq 3$, then $n(a_1 + a_3) > 3a_3 > 2a_2 + a_3 > 2a_2 + a_3 - a_1$. This is a contradiction. Therefore $n = 2$. If we multiply by 2 the sum of the first equation and the third one, we obtain

$$6a_1 + 2a_3 = 4a_2,$$

while the sum of the first one and the second one is

$$(m + 1)a_1 + (m - 1)a_2 = 2a_3.$$

Adding up the last two equations we get

$$(m + 7)a_1 = (5 - m)a_2.$$

It follows that $5 - m \geq 1$, because the left-hand side of the last equation and a_2 are positive. Since we have $m > n = 2$, the integer m can be equal only to either 3 or 4. Substituting $(3, 2)$ and $(4, 2)$ for (m, n) and solving the previous system of equations, we find the families of solutions $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer.

A2

Determine all sequences $(x_1, x_2, \dots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$

Answer. The only sequence that satisfies the condition is

$$(x_1, \dots, x_{2011}) = (1, k, \dots, k) \quad \text{with } k = 2 + 3 + \cdots + 2011 = 2023065.$$

Solution. Throughout this solution, the set of positive integers will be denoted by $\mathbb{Z}_+$.

Put $k = 2 + 3 + \cdots + 2011 = 2023065$. We have

$$1^n + 2k^n + \cdots 2011k^n = 1 + k \cdot k^n = k^{n+1} + 1$$

for all n, so $(1, k, \dots, k)$ is a valid sequence. We shall prove that it is the only one.

Let a valid sequence $(x_1, \dots, x_{2011})$ be given. For each $n \in \mathbb{Z}_+$ we have some $y_n \in \mathbb{Z}_+$ with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = y_n^{n+1} + 1.$$

Note that $x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n < (x_1 + 2x_2 + \cdots + 2011x_{2011})^{n+1}$, which implies that the sequence (y_n) is bounded. In particular, there is some $y \in \mathbb{Z}_+$ with $y_n = y$ for infinitely many n.

Let m be the maximum of all the x_i. Grouping terms with equal x_i together, the sum $x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n$ can be written as

$$x_1^n + 2x_2^n + \cdots + x_{2011}^n = a_m m^n + a_{m-1}(m-1)^n + \cdots + a_1$$

with $a_i \geq 0$ for all i and $a_1 + \cdots + a_m = 1 + 2 + \cdots + 2011$. So there exist arbitrarily large values of n, for which

$$a_m m^n + \cdots + a_1 - 1 - y \cdot y^n = 0. \tag{1}$$

The following lemma will help us to determine the a_i and y:

Lemma. Let integers $b_1, \dots, b_N$ be given and assume that there are arbitrarily large positive integers n with $b_1 + b_2 2^n + \cdots + b_N N^n = 0$. Then $b_i = 0$ for all i.

Proof. Suppose that not all b_i are zero. We may assume without loss of generality that $b_N \neq 0$.

Dividing through by N^n gives

$$|b_N| = \left| b_{N-1} \left(\frac{N-1}{N} \right)^n + \cdots + b_1 \left(\frac{1}{N} \right)^n \right| \leq (|b_{N-1}| + \cdots + |b_1|) \left(\frac{N-1}{N} \right)^n.$$

The expression $\left(\frac{N-1}{N}\right)^n$ can be made arbitrarily small for n large enough, contradicting the assumption that b_N be non-zero. $\square$

We obviously have $y > 1$. Applying the lemma to (1) we see that $a_m = y = m$, $a_1 = 1$, and all the other a_i are zero. This implies $(x_1, \ldots, x_{2011}) = (1, m, \ldots, m)$. But we also have $1 + m = a_1 + \cdots + a_m = 1 + \cdots + 2011 = 1 + k$ so $m = k$, which is what we wanted to show.

A3

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

Answer. Either both f and g vanish identically, or there exists a real number C such that $f(x) = x^2 + C$ and $g(x) = x$ for all real numbers x.

Solution. Clearly all these pairs of functions satisfy the functional equation in question, so it suffices to verify that there cannot be any further ones. Substituting $-2x$ for y in the given functional equation we obtain

$$g(f(-x)) = f(x). \tag{1}$$

Using this equation for $-x-y$ in place of x we obtain

$$f(-x-y) = g(f(x+y)) = f(x) + (2x+y)g(y). \tag{2}$$

Now for any two real numbers a and b, setting $x = -b$ and $y = a+b$ we get

$$f(-a) = f(-b) + (a-b)g(a+b).$$

If c denotes another arbitrary real number we have similarly

$$f(-b) = f(-c) + (b-c)g(b+c)$$

as well as

$$f(-c) = f(-a) + (c-a)g(c+a).$$

Adding all these equations up, we obtain

$$\big((a+c)-(b+c)\big)g(a+b) + \big((a+b)-(a+c)\big)g(b+c) + \big((b+c)-(a+b)\big)g(a+c) = 0.$$

Now given any three real numbers x, y, and z one may determine three reals a, b, and c such that $x = b+c$, $y = c+a$, and $z = a+b$, so that we get

$$(y-x)g(z) + (z-y)g(x) + (x-z)g(y) = 0.$$

This implies that the three points $(x, g(x))$, $(y, g(y))$, and $(z, g(z))$ from the graph of g are collinear. Hence that graph is a line, i.e., g is either a constant or a linear function.

Let us write $g(x) = Ax + B$, where A and B are two real numbers. Substituting $(0, -y)$ for (x, y) in (2) and denoting $C = f(0)$, we have $f(y) = Ay^2 - By + C$. Now, comparing the coefficients of x^2 in (1) we see that $A^2 = A$, so $A = 0$ or $A = 1$.

If $A = 0$, then (1) becomes $B = -Bx + C$ and thus $B = C = 0$, which provides the first of the two solutions mentioned above.

Now suppose $A = 1$. Then (1) becomes $x^2 - Bx + C + B = x^2 - Bx + C$, so $B = 0$. Thus, $g(x) = x$ and $f(x) = x^2 + C$, which is the second solution from above.

Comment. Another way to show that $g(x)$ is either a constant or a linear function is the following. If we interchange x and y in the given functional equation and subtract this new equation from the given one, we obtain

$$f(x) - f(y) = (2y + x)g(x) - (2x + y)g(y).$$

Substituting $(x, 0)$, $(1, x)$, and $(0, 1)$ for (x, y), we get

$$\begin{aligned}
f(x) - f(0) &= xg(x) - 2xg(0), \\
f(1) - f(x) &= (2x + 1)g(1) - (x + 2)g(x), \\
f(0) - f(1) &= 2g(0) - g(1).
\end{aligned}$$

Taking the sum of these three equations and dividing by 2, we obtain

$$g(x) = x\bigl(g(1) - g(0)\bigr) + g(0).$$

This proves that $g(x)$ is either a constant of a linear function.

A4

Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

Answer. The only pair (f, g) of functions that satisfies the equation is given by $f(n) = n$ and $g(n) = 1$ for all n.

Solution. The given relation implies

$$f\left(f^{g(n)}(n)\right) < f(n+1) \quad \text{for all } n, \tag{1}$$

which will turn out to be sufficient to determine f.

Let $y_1 < y_2 < \ldots$ be all the values attained by f (this sequence might be either finite or infinite). We will prove that for every positive n the function f attains at least n values, and we have (i)$_n$: $f(x) = y_n$ if and only if $x = n$, and (ii)$_n$: $y_n = n$. The proof will follow the scheme

$$\text{(i)}_1, \ \text{(ii)}_1, \ \text{(i)}_2, \ \text{(ii)}_2, \ \ldots, \ \text{(i)}_n, \ \text{(ii)}_n, \ \ldots \tag{2}$$

To start, consider any x such that $f(x) = y_1$. If $x > 1$, then (1) reads $f\big(f^{g(x-1)}(x-1)\big) < y_1$, contradicting the minimality of y_1. So we have that $f(x) = y_1$ is equivalent to $x = 1$, establishing (i)$_1$.

Next, assume that for some n statement (i)$_n$ is established, as well as all the previous statements in (2). Note that these statements imply that for all $k \geq 1$ and $a < n$ we have $f^k(x) = a$ if and only if $x = a$.

Now, each value y_i with $1 \leq i \leq n$ is attained at the unique integer i, so y_{n+1} exists. Choose an arbitrary x such that $f(x) = y_{n+1}$; we necessarily have $x > n$. Substituting $x - 1$ into (1) we have $f\big(f^{g(x-1)}(x-1)\big) < y_{n+1}$, which implies

$$f^{g(x-1)}(x-1) \in \{1, \ldots, n\} \tag{3}$$

Set $b = f^{g(x-1)}(x-1)$. If $b < n$ then we would have $x - 1 = b$ which contradicts $x > n$. So $b = n$, and hence $y_n = n$, which proves (ii)$_n$. Next, from (i)$_n$ we now get $f(k) = n \iff k = n$, so removing all the iterations of f in (3) we obtain $x - 1 = b = n$, which proves (i)$_{n+1}$.

So, all the statements in (2) are valid and hence $f(n) = n$ for all n. The given relation between f and g now reads $n + g^n(n) = n + 1 - g(n+1) + 1$ or $g^n(n) + g(n+1) = 2$, from which it immediately follows that we have $g(n) = 1$ for all n.

Comment. Several variations of the above solution are possible. For instance, one may first prove by induction that the smallest n values of f are exactly $f(1) < \cdots < f(n)$ and proceed as follows. We certainly have $f(n) \geq n$ for all n. If there is an n with $f(n) > n$, then $f(x) > x$ for all $x \geq n$. From this we conclude $f^{g(n)+1}(n) > f^{g(n)}(n) > \cdots > f(n)$. But we also have $f^{g(n)+1} < f(n+1)$. Having squeezed in a function value between $f(n)$ and $f(n+1)$, we arrive at a contradiction.

In any case, the inequality (1) plays an essential rôle.

A5

Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n+1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

Solution. Throughout the solution, we denote by $[a, b]$ the set $\{a, a+1, \ldots, b\}$. We say that $\{a, b, c\}$ is an *obtuse triple* if a, b, c are the sides of some obtuse triangle.

We prove by induction on n that there exists a partition of $[2, 3n+1]$ into n obtuse triples A_i $(2 \le i \le n+1)$ having the form $A_i = \{i, a_i, b_i\}$. For the base case $n = 1$, one can simply set $A_2 = \{2, 3, 4\}$. For the induction step, we need the following simple lemma.

Lemma. Suppose that the numbers $a < b < c$ form an obtuse triple, and let x be any positive number. Then the triple $\{a, b+x, c+x\}$ is also obtuse.

Proof. The numbers $a < b+x < c+x$ are the sides of a triangle because $(c+x)-(b+x) = c-b < a$. This triangle is obtuse since $(c+x)^2-(b+x)^2 = (c-b)(c+b+2x) > (c-b)(c+b) > a^2$. □

Now we turn to the induction step. Let $n > 1$ and put $t = \lfloor n/2 \rfloor < n$. By the induction hypothesis, there exists a partition of the set $[2, 3t+1]$ into t obtuse triples $A'_i = \{i, a'_i, b'_i\}$ $(i \in [2, t+1])$. For the same values of i, define $A_i = \{i, a'_i + (n-t), b'_i + (n-t)\}$. The constructed triples are obviously disjoint, and they are obtuse by the lemma. Moreover, we have

$$\bigcup_{i=2}^{t+1} A_i = [2, t+1] \cup [n+2, n+2t+1].$$

Next, for each $i \in [t+2, n+1]$, define $A_i = \{i, n+t+i, 2n+i\}$. All these sets are disjoint, and

$$\bigcup_{i=t+2}^{n+1} A_i = [t+2, n+1] \cup [n+2t+2, 2n+t+1] \cup [2n+t+2, 3n+1],$$

so

$$\bigcup_{i=2}^{n+1} A_i = [2, 3n+1].$$

Thus, we are left to prove that the triple A_i is obtuse for each $i \in [t+2, n+1]$.

Since $(2n+i)-(n+t+i) = n-t < t+2 \le i$, the elements of A_i are the sides of a triangle. Next, we have

$$(2n+i)^2-(n+t+i)^2 = (n-t)(3n+t+2i) \ge \frac{n}{2}\cdot(3n+3(t+1)+1) > \frac{n}{2}\cdot\frac{9n}{2} \ge (n+1)^2 \ge i^2,$$

so this triangle is obtuse. The proof is completed.

A6

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq yf(x) + f(f(x)) \tag{1}$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Solution 1. Substituting $y = t - x$, we rewrite (1) as

$$f(t) \leq tf(x) - xf(x) + f(f(x)). \tag{2}$$

Consider now some real numbers a, b and use (2) with $t = f(a)$, $x = b$ as well as with $t = f(b)$, $x = a$. We get

$$f(f(a)) - f(f(b)) \leq f(a)f(b) - bf(b),$$
$$f(f(b)) - f(f(a)) \leq f(a)f(b) - af(a).$$

Adding these two inequalities yields

$$2f(a)f(b) \geq af(a) + bf(b).$$

Now, substitute $b = 2f(a)$ to obtain $2f(a)f(b) \geq af(a) + 2f(a)f(b)$, or $af(a) \leq 0$. So, we get

$$f(a) \geq 0 \quad \text{for all } a < 0. \tag{3}$$

Now suppose $f(x) > 0$ for some real number x. From (2) we immediately get that for every $t < \dfrac{xf(x) - f(f(x))}{f(x)}$ we have $f(t) < 0$. This contradicts (3); therefore

$$f(x) \leq 0 \quad \text{for all real } x, \tag{4}$$

and by (3) again we get $f(x) = 0$ for all $x < 0$.

We are left to find $f(0)$. Setting $t = x < 0$ in (2) we get

$$0 \leq 0 - 0 + f(0),$$

so $f(0) \geq 0$. Combining this with (4) we obtain $f(0) = 0$.

Solution 2. We will also use the condition of the problem in form (2). For clarity we divide the argument into four steps.

Step 1. We begin by proving that f attains nonpositive values only. Assume that there exist some real number z with $f(z) > 0$. Substituting $x = z$ into (2) and setting $A = f(z)$, $B = -zf(z) - f(f(z))$ we get $f(t) \le At + B$ for all real t. Hence, if for any positive real number t we substitute $x = -t$, $y = t$ into (1), we get

$$\begin{aligned} f(0) &\le tf(-t) + f(f(-t)) \le t(-At + B) + Af(-t) + B \\ &\le -t(At - B) + A(-At + B) + B = -At^2 - (A^2 - B)t + (A + 1)B. \end{aligned}$$

But surely this cannot be true if we take t to be large enough. This contradiction proves that we have indeed $f(x) \le 0$ for all real numbers x. Note that for this reason (1) entails

$$f(x + y) \le yf(x) \tag{5}$$

for all real numbers x and y.

Step 2. We proceed by proving that f has at least one zero. If $f(0) = 0$, we are done. Otherwise, in view of Step 1 we get $f(0) < 0$. Observe that (5) tells us now $f(y) \le yf(0)$ for all real numbers y. Thus we can specify a positive real number a that is so large that $f(a)^2 > -f(0)$. Put $b = f(a)$ and substitute $x = b$ and $y = -b$ into (5); we learn $-b^2 < f(0) \le -bf(b)$, i.e. $b < f(b)$. Now we apply (2) to $x = b$ and $t = f(b)$, which yields

$$f(f(b)) \le \big(f(b) - b\big)f(b) + f(f(b)),$$

i.e. $f(b) \ge 0$. So in view of Step 1, b is a zero of f.

Step 3. Next we show that if $f(a) = 0$ and $b < a$, then $f(b) = 0$ as well. To see this, we just substitute $x = b$ and $y = a - b$ into (5), thus getting $f(b) \ge 0$, which suffices by Step 1.

Step 4. By Step 3, the solution of the problem is reduced to showing $f(0) = 0$. Pick any zero r of f and substitute $x = r$ and $y = -1$ into (1). Because of $f(r) = f(r - 1) = 0$ this gives $f(0) \ge 0$ and hence $f(0) = 0$ by Step 1 again.

Comment 1. Both of these solutions also show $f(x) \le 0$ for all real numbers x. As one can see from Solution 1, this task gets much easier if one already knows that f takes nonnegative values for sufficiently small arguments. Another way of arriving at this statement, suggested by the proposer, is as follows:

Put $a = f(0)$ and substitute $x = 0$ into (1). This gives $f(y) \le ay + f(a)$ for all real numbers y. Thus if for any real number x we plug $y = a - x$ into (1), we obtain

$$f(a) \le (a - x)f(x) + f(f(x)) \le (a - x)f(x) + af(x) + f(a)$$

and hence $0 \le (2a - x)f(x)$. In particular, if $x < 2a$, then $f(x) \ge 0$.

Having reached this point, one may proceed almost exactly as in the first solution to deduce $f(x) \le 0$ for all x. Afterwards the problem can be solved in a few lines as shown in steps 3 and 4 of the second solution.

Comment 2. The original problem also contained the question whether a nonzero function satisfying the problem condition exists. Here we present a family of such functions.

Notice first that if $g : (0, \infty) \longrightarrow [0, \infty)$ denotes any function such that

$$g(x + y) \geq yg(x) \tag{6}$$

for all positive real numbers x and y, then the function f given by

$$f(x) = \begin{cases} -g(x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{7}$$

automatically satisfies (1). Indeed, we have $f(x) \leq 0$ and hence also $f(f(x)) = 0$ for all real numbers x. So (1) reduces to (5); moreover, this inequality is nontrivial only if x and y are positive. In this last case it is provided by (6).

Now it is not hard to come up with a nonzero function g obeying (6). E.g. $g(z) = Ce^z$ (where C is a positive constant) fits since the inequality $e^y > y$ holds for all (positive) real numbers y. One may also consider the function $g(z) = e^z - 1$; in this case, we even have that f is continuous.

A7

Let a, b, and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}. \tag{1}$$

Throughout both solutions, we denote the sums of the form $f(a,b,c) + f(b,c,a) + f(c,a,b)$ by $\sum f(a,b,c)$.

Solution 1. The condition $b+c > \sqrt{2}$ implies $b^2+c^2 > 1$, so $a^2 = 3 - (b^2+c^2) < 2$, i.e. $a < \sqrt{2} < b+c$. Hence we have $b+c-a > 0$, and also $c+a-b > 0$ and $a+b-c > 0$ for similar reasons.

We will use the variant of Hölder's inequality

$$\frac{x_1^{p+1}}{y_1^p} + \frac{x_1^{p+1}}{y_1^p} + \ldots + \frac{x_n^{p+1}}{y_n^p} \geq \frac{(x_1+x_2+\ldots+x_n)^{p+1}}{(y_1+y_2+\ldots+y_n)^p},$$

which holds for all positive real numbers $p, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. Applying it to the left-hand side of (1) with $p = 2$ and $n = 3$, we get

$$\sum \frac{a}{(b+c-a)^2} = \sum \frac{(a^2)^3}{a^5(b+c-a)^2} \geq \frac{(a^2+b^2+c^2)^3}{\left(\sum a^{5/2}(b+c-a)\right)^2} = \frac{27}{\left(\sum a^{5/2}(b+c-a)\right)^2}. \tag{2}$$

To estimate the denominator of the right-hand part, we use an instance of Schur's inequality, namely

$$\sum a^{3/2}(a-b)(a-c) \geq 0,$$

which can be rewritten as

$$\sum a^{5/2}(b+c-a) \leq abc(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Moreover, by the inequality between the arithmetic mean and the fourth power mean we also have

$$\left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3}\right)^4 \leq \frac{a^2+b^2+c^2}{3} = 1,$$

i.e., $\sqrt{a}+\sqrt{b}+\sqrt{c} \leq 3$. Hence, (2) yields

$$\sum \frac{a}{(b+c-a)^2} \geq \frac{27}{\left(abc(\sqrt{a}+\sqrt{b}+\sqrt{c})\right)^2} \geq \frac{3}{a^2b^2c^2},$$

thus solving the problem.

Comment. In this solution, one may also start from the following version of HÖLDER's inequality

$$\left(\sum_{i=1}^{n} a_i^3\right)\left(\sum_{i=1}^{n} b_i^3\right)\left(\sum_{i=1}^{n} c_i^3\right) \geq \left(\sum_{i=1}^{n} a_i b_i c_i\right)^3$$

applied as

$$\sum \frac{a}{(b+c-a)^2} \cdot \sum a^3(b+c-a) \cdot \sum a^2(b+c-a) \geq 27.$$

After doing that, one only needs the slightly better known instances

$$\sum a^3(b+c-a) \leq (a+b+c)abc \qquad \text{and} \qquad \sum a^2(b+c-a) \leq 3abc$$

of SCHUR's Inequality.

Solution 2. As in Solution 1, we mention that all the numbers $b+c-a$, $a+c-b$, $a+b-c$ are positive. We will use only this restriction and the condition

$$a^5+b^5+c^5 \geq 3, \tag{3}$$

which is weaker than the given one. Due to the symmetry we may assume that $a \geq b \geq c$.

In view of (3), it suffices to prove the inequality

$$\sum \frac{a^3b^2c^2}{(b+c-a)^2} \geq \sum a^5,$$

or, moving all the terms into the left-hand part,

$$\sum \frac{a^3}{(b+c-a)^2}\left((bc)^2-(a(b+c-a))^2\right) \geq 0. \tag{4}$$

Note that the signs of the expressions $(yz)^2-(x(y+z-x))^2$ and $yz-x(y+z-x)=(x-y)(x-z)$ are the same for every positive x, y, z satisfying the triangle inequality. So the terms in (4) corresponding to a and c are nonnegative, and hence it is sufficient to prove that the sum of the terms corresponding to a and b is nonnegative. Equivalently, we need the relation

$$\frac{a^3}{(b+c-a)^2}(a-b)(a-c)(bc+a(b+c-a)) \geq \frac{b^3}{(a+c-b)^2}(a-b)(b-c)(ac+b(a+c-b)).$$

Obviously, we have

$$a^3 \geq b^3 \geq 0, \quad 0 < b+c-a \leq a+c-b, \quad \text{and} \quad a-c \geq b-c \geq 0,$$

hence it suffices to prove that

$$\frac{ab+ac+bc-a^2}{b+c-a} \geq \frac{ab+ac+bc-b^2}{c+a-b}.$$

Since all the denominators are positive, it is equivalent to

$$(c+a-b)(ab+ac+bc-a^2)-(ab+ac+bc-b^2)(b+c-a)\geq 0,$$

or

$$(a-b)(2ab-a^2-b^2+ac+bc)\geq 0.$$

Since $a\geq b$, the last inequality follows from

$$c(a+b)>(a-b)^2$$

which holds since $c>a-b\geq 0$ and $a+b>a-b\geq 0$.

C1

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

Answer. The number $f(n)$ of ways of placing the n weights is equal to the product of all odd positive integers less than or equal to $2n-1$, i.e. $f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$.

Solution 1. Assume $n \geq 2$. We claim

$$f(n) = (2n-1)f(n-1). \tag{1}$$

Firstly, note that after the first move the left pan is always at least 1 heavier than the right one. Hence, any valid way of placing the n weights on the scale gives rise, by not considering weight 1, to a valid way of placing the weights $2, 2^2, \ldots, 2^{n-1}$.

If we divide the weight of each weight by 2, the answer does not change. So these $n-1$ weights can be placed on the scale in $f(n-1)$ valid ways. Now we look at weight 1. If it is put on the scale in the first move, then it has to be placed on the left side, otherwise it can be placed either on the left or on the right side, because after the first move the difference between the weights on the left pan and the weights on the right pan is at least 2. Hence, there are exactly $2n-1$ different ways of inserting weight 1 in each of the $f(n-1)$ valid sequences for the $n-1$ weights in order to get a valid sequence for the n weights. This proves the claim.

Since $f(1) = 1$, by induction we obtain for all positive integers n

$$f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

Comment 1. The word "compute" in the statement of the problem is probably too vague. An alternative but more artificial question might ask for the smallest n for which the number of valid ways is divisible by 2011. In this case the answer would be 1006.

Comment 2. It is useful to remark that the answer is the same for any set of weights where each weight is heavier than the sum of the lighter ones. Indeed, in such cases the given condition is equivalent to asking that during the process the heaviest weight on the balance is always on the left pan.

Comment 3. Instead of considering the lightest weight, one may also consider the last weight put on the balance. If this weight is 2^{n-1} then it should be put on the left pan. Otherwise it may be put on any pan; the inequality would not be violated since at this moment the heaviest weight is already put onto the left pan. In view of the previous comment, in each of these $2n-1$ cases the number of ways to place the previous weights is exactly $f(n-1)$, which yields (1).

Solution 2. We present a different way of obtaining (1). Set $f(0)=1$. Firstly, we find a recurrent formula for $f(n)$.

Assume $n \geq 1$. Suppose that weight 2^{n-1} is placed on the balance in the i-th move with $1 \leq i \leq n$. This weight has to be put on the left pan. For the previous moves we have $\binom{n-1}{i-1}$ choices of the weights and from Comment 2 there are $f(i-1)$ valid ways of placing them on the balance. For later moves there is no restriction on the way in which the weights are to be put on the pans. Therefore, all $(n-i)!2^{n-i}$ ways are possible. This gives

$$f(n)=\sum_{i=1}^{n}\binom{n-1}{i-1}f(i-1)(n-i)!2^{n-i}=\sum_{i=1}^{n}\frac{(n-1)!f(i-1)2^{n-i}}{(i-1)!}. \qquad (2)$$

Now we are ready to prove (1). Using $n-1$ instead of n in (2) we get

$$f(n-1)=\sum_{i=1}^{n-1}\frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}.$$

Hence, again from (2) we get

$$\begin{aligned}f(n)&=2(n-1)\sum_{i=1}^{n-1}\frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}+f(n-1)\\&=(2n-2)f(n-1)+f(n-1)=(2n-1)f(n-1),\end{aligned}$$

QED.

Comment. There exist different ways of obtaining the formula (2). Here we show one of them.

Suppose that in the first move we use weight 2^{n-i+1}. Then the lighter $n-i$ weights may be put on the balance at any moment and on either pan. This gives $2^{n-i}\cdot(n-1)!/(i-1)!$ choices for the moves (moments and choices of pan) with the lighter weights. The remaining $i-1$ moves give a valid sequence for the $i-1$ heavier weights and this is the only requirement for these moves, so there are $f(i-1)$ such sequences. Summing over all $i=1,2,\ldots,n$ we again come to (2).

C2

Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

Solution. Number the students consecutively from 1 to 1000. Let $a_i = 1$ if the ith student is a girl, and $a_i = 0$ otherwise. We expand this notion for all integers i by setting $a_{i+1000} = a_{i-1000} = a_i$. Next, let

$$S_k(i) = a_i + a_{i+1} + \cdots + a_{i+k-1}.$$

Now the statement of the problem can be reformulated as follows:

There exist an integer k with $100 \leq k \leq 300$ and an index i such that $S_k(i) = S_k(i+k)$.

Assume now that this statement is false. Choose an index i such that $S_{100}(i)$ attains the maximal possible value. In particular, we have $S_{100}(i-100) - S_{100}(i) < 0$ and $S_{100}(i) - S_{100}(i+100) > 0$, for if we had an equality, then the statement would hold. This means that the function $S(j) - S(j+100)$ changes sign somewhere on the segment $[i-100, i]$, so there exists some index $j \in [i-100, i-1]$ such that

$$S_{100}(j) \leq S_{100}(j+100) - 1, \quad \text{but} \quad S_{100}(j+1) \geq S_{100}(j+101) + 1. \tag{1}$$

Subtracting the first inequality from the second one, we get $a_{j+100} - a_j \geq a_{j+200} - a_{j+100} + 2$, so

$$a_j = 0, \quad a_{j+100} = 1, \quad a_{j+200} = 0.$$

Substituting this into the inequalities of (1), we also obtain $S_{99}(j+1) \leq S_{99}(j+101) \leq S_{99}(j+1)$, which implies

$$S_{99}(j+1) = S_{99}(j+101). \tag{2}$$

Now let k and ℓ be the least positive integers such that $a_{j-k} = 1$ and $a_{j+200+\ell} = 1$. By symmetry, we may assume that $k \geq \ell$. If $k \geq 200$ then we have $a_j = a_{j-1} = \cdots = a_{j-199} = 0$, so $S_{100}(j-199) = S_{100}(j-99) = 0$, which contradicts the initial assumption. Hence $\ell \leq k \leq 199$.

Finally, we have

$$S_{100+\ell}(j-\ell+1) = (a_{j-\ell+1} + \cdots + a_j) + S_{99}(j+1) + a_{j+100} = S_{99}(j+1) + 1,$$
$$S_{100+\ell}(j+101) = S_{99}(j+101) + (a_{j+200} + \cdots + a_{j+200+\ell-1}) + a_{j+200+\ell} = S_{99}(j+101) + 1.$$

Comparing with (2) we get $S_{100+\ell}(j-\ell+1) = S_{100+\ell}(j+101)$ and $100 + \ell \leq 299$, which again contradicts our assumption.

Comment. It may be seen from the solution that the number 300 from the problem statement can be replaced by 299. Here we consider some improvements of this result. Namely, we investigate which interval can be put instead of $[100, 300]$ in order to keep the problem statement valid.

First of all, the two examples

$$\underbrace{1,1,\ldots,1}_{167},\underbrace{0,0,\ldots,0}_{167},\underbrace{1,1,\ldots,1}_{167},\underbrace{0,0,\ldots,0}_{167},\underbrace{1,1,\ldots,1}_{167},\underbrace{0,0,\ldots,0}_{165}$$

and

$$\underbrace{1,1,\ldots,1}_{249},\underbrace{0,0,\ldots,0}_{251},\underbrace{1,1,\ldots,1}_{249},\underbrace{0,0,\ldots,0}_{251}$$

show that the interval can be changed neither to $[84, 248]$ nor to $[126, 374]$.

On the other hand, we claim that this interval can be changed to $[125, 250]$. Note that this statement is invariant under replacing all 1's by 0's and vice versa. Assume, to the contrary, that there is no admissible $k \in [125, 250]$. The arguments from the solution easily yield the following lemma.

Lemma. Under our assumption, suppose that for some indices $i < j$ we have $S_{125}(i) \le S_{125}(i + 125)$ but $S_{125}(j) \ge S_{125}(j+125)$. Then there exists some $t \in [i, j-1]$ such that $a_t = a_{t-1} = \cdots = a_{t-125} = 0$ and $a_{t+250} = a_{t+251} = \cdots = a_{t+375} = 0$. □

Let us call a segment $[i, j]$ of indices a *crowd*, if (a) $a_i = a_{i+1} = \cdots = a_j$, but $a_{i-1} \ne a_i \ne a_{j+1}$, and (b) $j - i \ge 125$. Now, using the lemma, one can get in the same way as in the solution that there exists some crowd. Take all the crowds in the circle, and enumerate them in cyclic order as $A_1, \ldots, A_d$. We also assume always that $A_{s+d} = A_{s-d} = A_s$.

Consider one of the crowds, say A_1. We have $A_1 = [i, i + t]$ with $125 \le t \le 248$ (if $t \ge 249$, then $a_i = a_{i+1} = \cdots = a_{i+249}$ and therefore $S_{125}(i) = S_{125}(i + 125)$, which contradicts our assumption). We may assume that $a_i = 1$. Then we have $S_{125}(i + t - 249) \le 125 = S_{125}(i + t - 124)$ and $S_{125}(i) = 125 \ge S_{125}(i + 125)$, so by the lemma there exists some index $j \in [i + t - 249, i - 1]$ such that the segments $[j - 125, j]$ and $[j + 250, j + 375]$ are contained in some crowds.

Let us fix such j and denote the segment $[j + 1, j + 249]$ by B_1. Clearly, $A_1 \subseteq B_1$. Moreover, B_1 cannot contain any crowd other than A_1 since $|B_1| = 249 < 2 \cdot 126$. Hence it is clear that $j \in A_d$ and $j + 250 \in A_2$. In particular, this means that the genders of A_d and A_2 are different from that of A_1.

Performing this procedure for every crowd A_s, we find segments $B_s = [j_s + 1, j_s + 249]$ such that $|B_s| = 249$, $A_s \subseteq B_s$, and $j_s \in A_{s-1}$, $j_s + 250 \in A_{s+1}$. So, B_s covers the whole segment between A_{s-1} and A_{s+1}, hence the sets $B_1, \ldots, B_d$ cover some 1000 consecutive indices. This implies $249d \ge 1000$, and $d \ge 5$. Moreover, the gender of A_i is alternating, so d is even; therefore $d \ge 6$.

Consider now three segments $A_1 = [i_1, i'_1]$, $B_2 = [j_2 + 1, j_2 + 249]$, $A_3 = [i_3, i'_3]$. By construction, we have $[j_2 - 125, j_2] \subseteq A_1$ and $[j_2 + 250, j_2 + 375] \subseteq A_3$, whence $i_1 \le j_2 - 125$, $i'_3 \ge j_2 + 375$. Therefore $i'_3 - i_1 \ge 500$. Analogously, if $A_4 = [i_4, i'_4]$, $A_6 = [i_6, i'_6]$ then $i'_6 - i_4 \ge 500$. But from $d \ge 6$ we get $i_1 < i'_3 < i_4 < i'_6 < i_1 + 1000$, so $1000 > (i'_3 - i_1) + (i'_6 - i_4) \ge 500 + 500$. This final contradiction shows that our claim holds.

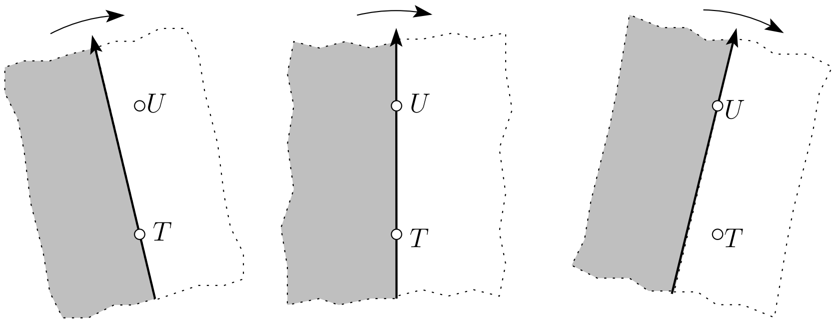
One may even show that the interval in the statement of the problem may be replaced by $[125, 249]$ (both these numbers cannot be improved due to the examples above). But a proof of this fact is a bit messy, and we do not present it here.

C3

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

Solution. Give the rotating line an orientation and distinguish its sides as the *oranje side* and the *blue side*. Notice that whenever the pivot changes from some point T to another point U, after the change, T is on the same side as U was before. Therefore, the number of elements of $\mathcal{S}$ on the oranje side and the number of those on the blue side remain the same throughout the whole process (except for those moments when the line contains two points).



First consider the case that $|\mathcal{S}| = 2n+1$ is odd. We claim that through any point $T \in \mathcal{S}$, there is a line that has n points on each side. To see this, choose an oriented line through T containing no other point of $\mathcal{S}$ and suppose that it has $n+r$ points on its oranje side. If $r = 0$ then we have established the claim, so we may assume that $r \neq 0$. As the line rotates through $180°$ around T, the number of points of $\mathcal{S}$ on its oranje side changes by 1 whenever the line passes through a point; after $180°$, the number of points on the oranje side is $n - r$. Therefore there is an intermediate stage at which the oranje side, and thus also the blue side, contains n points.

Now select the point P arbitrarily, and choose a line through P that has n points of $\mathcal{S}$ on each side to be the initial state of the windmill. We will show that during a rotation over $180°$, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and select a line ℓ through T that separates $\mathcal{S}$ into equal halves. The point T is the unique point of $\mathcal{S}$ through which a line in this direction can separate the points of $\mathcal{S}$ into equal halves (parallel translation would disturb the balance). Therefore, when the windmill line is parallel to ℓ, it must be ℓ itself, and so pass through T.

Next suppose that $|\mathcal{S}| = 2n$. Similarly to the odd case, for every $T \in \mathcal{S}$ there is an oriented line through T with $n-1$ points on its oranje side and n points on its blue side. Select such an oriented line through an arbitrary P to be the initial state of the windmill.

We will now show that during a rotation over 360°, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and an oriented line ℓ through T that separates $\mathcal{S}$ into two subsets with $n-1$ points on its oranje and n points on its blue side. Again, parallel translation would change the numbers of points on the two sides, so when the windmill line is parallel to ℓ with the same orientation, the windmill line must pass through T.

Comment. One may shorten this solution in the following way.

Suppose that $|\mathcal{S}| = 2n+1$. Consider any line ℓ that separates $\mathcal{S}$ into equal halves; this line is unique given its direction and contains some point $T \in \mathcal{S}$. Consider the windmill starting from this line. When the line has made a rotation of 180°, it returns to the same location but the oranje side becomes blue and vice versa. So, for each point there should have been a moment when it appeared as pivot, as this is the only way for a point to pass from on side to the other.

Now suppose that $|\mathcal{S}| = 2n$. Consider a line having $n-1$ and n points on the two sides; it contains some point T. Consider the windmill starting from this line. After having made a rotation of 180°, the windmill line contains some different point R, and each point different from T and R has changed the color of its side. So, the windmill should have passed through all the points.

C4

Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

Answer. The greatest such number k is 3.

Solution 1. There are various examples showing that $k = 3$ does indeed have the property under consideration. E.g. one can take

$$\begin{aligned} A_1 &= \{1, 2, 3\} \cup \{3m \mid m \geq 4\}, \\ A_2 &= \{4, 5, 6\} \cup \{3m - 1 \mid m \geq 4\}, \\ A_3 &= \{7, 8, 9\} \cup \{3m - 2 \mid m \geq 4\}. \end{aligned}$$

To check that this partition fits, we notice first that the sums of two distinct elements of A_i obviously represent all numbers $n \geq 1 + 12 = 13$ for $i = 1$, all numbers $n \geq 4 + 11 = 15$ for $i = 2$, and all numbers $n \geq 7 + 10 = 17$ for $i = 3$. So, we are left to find representations of the numbers 15 and 16 as sums of two distinct elements of A_3. These are $15 = 7+8$ and $16 = 7+9$.

Let us now suppose that for some $k \geq 4$ there exist sets $A_1, A_2, \ldots, A_k$ satisfying the given property. Obviously, the sets $A_1, A_2, A_3, A_4 \cup \cdots \cup A_k$ also satisfy the same property, so one may assume $k = 4$.

Put $B_i = A_i \cap \{1, 2, \ldots, 23\}$ for $i = 1, 2, 3, 4$. Now for any index i each of the ten numbers $15, 16, \ldots, 24$ can be written as sum of two distinct elements of B_i. Therefore this set needs to contain at least five elements. As we also have $|B_1| + |B_2| + |B_3| + |B_4| = 23$, there has to be some index j for which $|B_j| = 5$. Let $B_j = \{x_1, x_2, x_3, x_4, x_5\}$. Finally, now the sums of two distinct elements of A_j representing the numbers $15, 16, \ldots, 24$ should be exactly all the pairwise sums of the elements of B_j. Calculating the sum of these numbers in two different ways, we reach

$$4(x_1 + x_2 + x_3 + x_4 + x_5) = 15 + 16 + \ldots + 24 = 195.$$

Thus the number 195 should be divisible by 4, which is false. This contradiction completes our solution.

Comment. There are several variation of the proof that k should not exceed 3. E.g., one may consider the sets $C_i = A_i \cap \{1, 2, \ldots, 19\}$ for $i = 1, 2, 3, 4$. As in the previous solution one can show that for some index j one has $|C_j| = 4$, and the six pairwise sums of the elements of C_j should represent all numbers $15, 16, \ldots, 20$. Let $C_j = \{y_1, y_2, y_3, y_4\}$ with $y_1 < y_2 < y_3 < y_4$. It is not hard to deduce

$C_j = \{7, 8, 9, 11\}$, so in particular we have $1 \notin C_j$. Hence it is impossible to represent 21 as sum of two distinct elements of A_j, which completes our argument.

Solution 2. Again we only prove that $k \leq 3$. Assume that $A_1, A_2, \ldots, A_k$ is a partition satisfying the given property. We construct a graph $\mathcal{G}$ on the set $V = \{1, 2, \ldots, 18\}$ of *vertices* as follows. For each $i \in \{1, 2, \ldots, k\}$ and each $d \in \{15, 16, 17, 19\}$ we choose one pair of distinct elements $a, b \in A_i$ with $a + b = d$, and we draw an *edge* in the i^{th} color connecting a with b. By hypothesis, $\mathcal{G}$ has exactly 4 edges of each color.

Claim. The graph $\mathcal{G}$ contains at most one circuit.

Proof. Note that all the connected components of $\mathcal{G}$ are monochromatic and hence contain at most four edges. Thus also all circuits of $\mathcal{G}$ are monochromatic and have length at most four. Moreover, each component contains at most one circuit since otherwise it should contain at least five edges.

Suppose that there is a 4-cycle in $\mathcal{G}$, say with vertices a, b, c, and d in order. Then $\{a+b, b+c, c+d, d+a\} = \{15, 16, 17, 19\}$. Taking sums we get $2(a+b+c+d) = 15+16+17+19$ which is impossible for parity reasons. Thus all circuits of $\mathcal{G}$ are triangles.

Now if the vertices a, b, and c form such a triangle, then by a similar reasoning the set $\{a+b, b+c, c+a\}$ coincides with either $\{15, 16, 17\}$, or $\{15, 16, 19\}$, or $\{16, 17, 19\}$, or $\{15, 17, 19\}$. The last of these alternatives can be excluded for parity reasons again, whilst in the first three cases the set $\{a, b, c\}$ appears to be either $\{7, 8, 9\}$, or $\{6, 9, 10\}$, or $\{7, 9, 10\}$, respectively. Thus, a component containing a circuit should contain 9 as a vertex. Therefore there is at most one such component and hence at most one circuit. $\square$

By now we know that $\mathcal{G}$ is a graph with $4k$ edges, at least k components and at most one circuit. Consequently, $\mathcal{G}$ must have at least $4k + k - 1$ vertices. Thus $5k - 1 \leq 18$, and $k \leq 3$.

C5

Let m be a positive integer and consider a checkerboard consisting of m by m unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn $90°$ clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard or prove that such a moment does not necessarily exist.

Antswer. The latest possible moment for the last ant to fall off is $\frac{3m}{2} - 1$.

Solution. For $m = 1$ the answer is clearly correct, so assume $m > 1$. In the sequel, the word *collision* will be used to denote meeting of exactly two ants, moving in opposite directions.

If at the beginning we place an ant on the southwest corner square facing east and an ant on the southeast corner square facing west, then they will meet in the middle of the bottom row at time $\frac{m-1}{2}$. After the collision, the ant that moves to the north will stay on the board for another $m - \frac{1}{2}$ time units and thus we have established an example in which the last ant falls off at time $\frac{m-1}{2} + m - \frac{1}{2} = \frac{3m}{2} - 1$. So, we are left to prove that this is the latest possible moment.

Consider any collision of two ants a and a'. Let us change the rule for this collision, and enforce these two ants to turn anticlockwise. Then the succeeding behavior of all the ants does not change; the only difference is that a and a' swap their positions. These arguments may be applied to any collision separately, so we may assume that at any collision, either both ants rotate clockwise or both of them rotate anticlockwise by our own choice.

For instance, we may assume that there are only two types of ants, depending on their initial direction: *NE-ants*, which move only north or east, and *SW-ants*, moving only south and west. Then we immediately obtain that all ants will have fallen off the board after $2m - 1$ time units. However, we can get a better bound by considering the last moment at which a given ant collides with another ant.

Choose a coordinate system such that the corners of the checkerboard are $(0, 0)$, $(m, 0)$, (m, m) and $(0, m)$. At time t, there will be no NE-ants in the region $\{(x, y) : x + y < t + 1\}$ and no SW-ants in the region $\{(x, y) : x + y > 2m - t - 1\}$. So if two ants collide at (x, y) at time t, we have

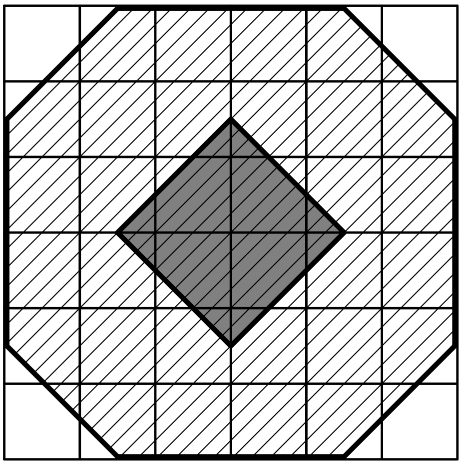
$$t + 1 \le x + y \le 2m - t - 1. \tag{1}$$

Analogously, we may change the rules so that each ant would move either alternatingly north and west, or alternatingly south and east. By doing so, we find that apart from (1) we also have $|x-y| \le m-t-1$ for each collision at point (x,y) and time t.

To visualize this, put

$$B(t) = \left\{(x,y) \in [0,m]^2 \colon t+1 \le x+y \le 2m-t-1 \text{ and } |x-y| \le m-t-1\right\}.$$

An ant can thus only collide with another ant at time t if it happens to be in the region $B(t)$. The following figure displays $B(t)$ for $t=\frac{1}{2}$ and $t=\frac{7}{2}$ in the case $m=6$:



Now suppose that an NE-ant has its last collision at time t and that it does so at the point (x,y) (if the ant does not collide at all, it will fall off the board within $m-\frac{1}{2} < \frac{3m}{2}-1$ time units, so this case can be ignored). Then we have $(x,y) \in B(t)$ and thus $x+y \ge t+1$ and $x-y \ge -(m-t-1)$. So we get

$$x \ge \frac{(t+1)-(m-t-1)}{2} = t+1-\frac{m}{2}.$$

By symmetry we also have $y \ge t+1-\frac{m}{2}$, and hence $\min\{x,y\} \ge t+1-\frac{m}{2}$. After this collision, the ant will move directly to an edge, which will take at most $m-\min\{x,y\}$ units of time. In sum, the total amount of time the ant stays on the board is at most

$$t+(m-\min\{x,y\}) \le t+m-\left(t+1-\frac{m}{2}\right) = \frac{3m}{2}-1.$$

By symmetry, the same bound holds for SW-ants as well.

C6

Let n be a positive integer and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to *appear* in W if there exist indices $k \le \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

Solution. Throughout the solution, all the words are nonempty. For any word R of length m, we call the number of indices $i \in \{1, 2, \ldots, N\}$ for which R coincides with the subword $x_{i+1}x_{i+2}\ldots x_{i+m}$ of W the *multiplicity* of R and denote it by $\mu(R)$. Thus a word R appears in W if and only if $\mu(R) > 0$. Since each occurrence of a word in W is both succeeded by either the letter a or the letter b and similarly preceded by one of those two letters, we have

$$\mu(R) = \mu(Ra) + \mu(Rb) = \mu(aR) + \mu(bR) \tag{1}$$

for all words R.

We claim that the condition that N is in fact the minimal period of W guarantees that each word of length N has multiplicity 1 or 0 depending on whether it appears or not. Indeed, if the words $x_{i+1}x_{i+2}\ldots x_{i+N}$ and $x_{j+1}\ldots x_{j+N}$ are equal for some $1 \le i < j \le N$, then we have $x_{i+a} = x_{j+a}$ for every integer a, and hence $j - i$ is also a period.

Moreover, since $N > 2^n$, at least one of the two words a and b has a multiplicity that is strictly larger than 2^{n-1}.

For each $k = 0, 1, \ldots, n-1$, let U_k be a subword of W whose multiplicity is strictly larger than 2^k and whose length is maximal subject to this property. Note that such a word exists in view of the two observations made in the two previous paragraphs.

Fix some index $k \in \{0, 1, \ldots, n-1\}$. Since the word U_kb is longer than U_k, its multiplicity can be at most 2^k, so in particular $\mu(U_kb) < \mu(U_k)$. Therefore, the word U_ka has to appear by (1). For a similar reason, the words U_kb, aU_k, and bU_k have to appear as well. Hence, the word U_k is ubiquitous. Moreover, if the multiplicity of U_k were strictly greater than 2^{k+1}, then by (1) at least one of the two words U_ka and U_kb would have multiplicity greater than 2^k and would thus violate the maximality condition imposed on U_k.

So we have $\mu(U_0) \le 2 < \mu(U_1) \le 4 < \ldots \le 2^{n-1} < \mu(U_{n-1})$, which implies in particular that the words $U_0, U_1, \ldots, U_{n-1}$ have to be distinct. As they have been proved to be ubiquitous as well, the problem is solved.

Comment 1. There is an easy construction for obtaining ubiquitous words from appearing words whose multiplicity is at least two. Starting with any such word U we may simply extend one of its occurrences in W forwards and backwards as long as its multiplicity remains fixed, thus arriving at a word that one might call the *ubiquitous prolongation* $p(U)$ of U.

There are several variants of the argument in the second half of the solution using the concept of prolongation. For instance, one may just take all ubiquitous words $U_1, U_2, \ldots, U_\ell$ ordered by increasing multiplicity and then prove for $i \in \{1, 2, \ldots, \ell\}$ that $\mu(U_i) \leq 2^i$. Indeed, assume that i is a minimal counterexample to this statement; then by the arguments similar to those presented above, the ubiquitous prolongation of one of the words $U_i a$, $U_i b$, aU_i or bU_i violates the definition of U_i.

Now the multiplicity of one of the two letters a and b is strictly greater than 2^{n-1}, so passing to ubiquitous prolongations once more we obtain $2^{n-1} < \mu(U_\ell) \leq 2^\ell$, which entails $\ell \geq n$, as needed.

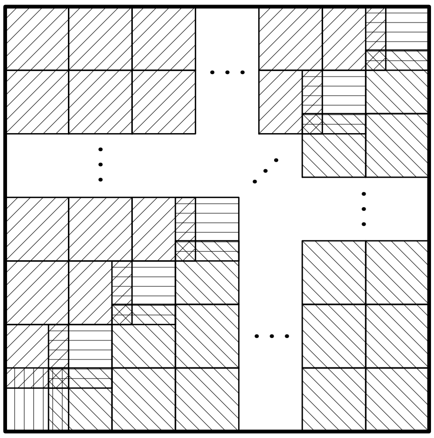
Comment 2. The bound n for the number of ubiquitous subwords in the problem statement is not optimal, but it is close to an optimal one in the following sense. There is a universal constant $C > 0$ such that for each positive integer n there exists an infinite periodic word W whose minimal period is greater than 2^n but for which there exist fewer than Cn ubiquitous words.

C7

On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

Answer. $2011^2 - \big((52^2 - 35^2)\cdot 39 - 17^2\big) = 4044121 - 57392 = 3986729$.

Solution 1. Let $m = 39$, then $2011 = 52m - 17$. We begin with an example showing that there can exist 3986729 cells carrying the same positive number.



To describe it, we number the columns from the left to the right and the rows from the bottom to the top by $1, 2, \dots, 2011$. We will denote each napkin by the coordinates of its lower-left cell. There are four kinds of napkins: first, we take all napkins $(52i + 36, 52j + 1)$ with $0 \le j \le i \le m - 2$; second, we use all napkins $(52i + 1, 52j + 36)$ with $0 \le i \le j \le m - 2$; third, we use all napkins $(52i + 36, 52i + 36)$ with $0 \le i \le m - 2$; and finally the napkin $(1, 1)$. Different groups of napkins are shown by different types of hatchings in the picture.

Now except for those squares that carry two or more different hatchings, all squares have the number 1 written into them. The number of these exceptional cells is easily computed to be $(52^2 - 35^2)m - 17^2 = 57392$.

We are left to prove that 3986729 is an upper bound for the number of cells containing the same number. Consider any configuration of napkins and any positive integer M. Suppose there are g cells with a number different from M. Then it suffices to show $g \ge 57392$. Throughout the solution, a *line* will mean either a row or a column.

Consider any line ℓ. Let $a_1, \dots, a_{52m-17}$ be the numbers written into its consecutive cells. For $i = 1, 2, \dots, 52$, let $s_i = \sum_{t \equiv i \pmod{52}} a_t$. Note that $s_1, \dots, s_{35}$ have m terms each, while $s_{36}, \dots, s_{52}$ have $m-1$ terms each. Every napkin intersecting ℓ contributes exactly 1 to each s_i;

hence the number s of all those napkins satisfies $s_1 = \cdots = s_{52} = s$. Call the line ℓ *rich* if $s > (m-1)M$ and *poor* otherwise.

Suppose now that ℓ is rich. Then in each of the sums $s_{36}, \ldots, s_{52}$ there exists a term greater than M; consider all these terms and call the corresponding cells the *rich bad cells* for this line. So, each rich line contains at least 17 cells that are bad for this line.

If, on the other hand, ℓ is poor, then certainly $s < mM$ so in each of the sums $s_1, \ldots, s_{35}$ there exists a term less than M; consider all these terms and call the corresponding cells the *poor bad cells* for this line. So, each poor line contains at least 35 cells that are bad for this line.

Let us call all indices congruent to $1, 2, \ldots,$ or 35 modulo 52 *small*, and all other indices, i.e. those congruent to $36, 37, \ldots,$ or 52 modulo 52, *big*. Recall that we have numbered the columns from the left to the right and the rows from the bottom to the top using the numbers $1, 2, \ldots, 52m - 17$; we say that a line is *big* or *small* depending on whether its index is big or small. By definition, all rich bad cells for the rows belong to the big columns, while the poor ones belong to the small columns, and vice versa.

In each line, we put a *strawberry* on each cell that is bad for this line. In addition, for each **small rich** line we put an *extra* strawberry on each of its (rich) bad cells. A cell gets the strawberries from its row and its column independently.

Notice now that a cell with a strawberry on it contains a number different from M. If this cell gets a strawberry by the extra rule, then it contains a number greater than M. Moreover, it is either in a small row and in a big column, or vice versa. Suppose that it is in a small row, then it is not bad for its column. So it has not more than two strawberries in this case. On the other hand, if the extra rule is not applied to some cell, then it also has not more than two strawberries. So, the total number N of strawberries is at most $2g$.

We shall now estimate N in a different way. For each of the $2 \cdot 35m$ small lines, we have introduced at least 34 strawberries if it is rich and at least 35 strawberries if it is poor, so at least 34 strawberries in any case. Similarly, for each of the $2 \cdot 17(m-1)$ big lines, we put at least $\min(17, 35) = 17$ strawberries. Summing over all lines we obtain

$$2g \geq N \geq 2(35m \cdot 34 + 17(m-1) \cdot 17) = 2(1479m - 289) = 2 \cdot 57392,$$

as desired.

Comment. The same reasoning applies also if we replace 52 by R and 2011 by $Rm - H$, where m, R, and H are integers with $m, R \geq 1$ and $0 \leq H \leq \frac{1}{3}R$. More detailed information is provided after the next solution.

Solution 2. We present a different proof of the estimate which is the hard part of the problem. Let $S = 35$, $H = 17$, $m = 39$; so the table size is $2011 = Sm + H(m-1)$, and the napkin size is $52 = S + H$. Fix any positive integer M and call a cell *vicious* if it contains a number distinct from M. We will prove that there are at least $H^2(m-1)+2SHm$ vicious cells.

Firstly, we introduce some terminology. As in the previous solution, we number rows and columns and we use the same notions of *small* and *big* indices and lines; so, an index is small if it is congruent to one of the numbers $1, 2, \dots, S$ modulo $(S+H)$. The numbers $1, 2, \dots, S+H$ will be known as *residues*. For two residues i and j, we say that a cell is of *type* (i, j) if the index of its row is congruent to i and the index of its column to j modulo $(S+H)$. The number of vicious cells of this type is denoted by v_{ij}.

Let s, s' be two variables ranging over small residues and let h, h' be two variables ranging over big residues. A cell is said to be of *class* A, B, C, or D if its type is of shape (s, s'), (s, h), (h, s), or (h, h'), respectively. The numbers of vicious cells belonging to these classes are denoted in this order by a, b, c, and d. Observe that each cell belongs to exactly one class.

Claim 1. We have

$$m \leq \frac{a}{S^2} + \frac{b+c}{2SH}. \tag{1}$$

Proof. Consider an arbitrary small row r. Denote the numbers of vicious cells on r belonging to the classes A and B by α and β, respectively. As in the previous solution, we obtain that $\alpha \geq S$ or $\beta \geq H$. So in each case we have $\frac{\alpha}{S} + \frac{\beta}{H} \geq 1$.

Performing this argument separately for each small row and adding up all the obtained inequalities, we get $\frac{a}{S} + \frac{b}{H} \geq mS$. Interchanging rows and columns we similarly get $\frac{a}{S} + \frac{c}{H} \geq mS$. Summing these inequalities and dividing by $2S$ we get what we have claimed. $\square$

Claim 2. Fix two small residue s, s' and two big residues h, h'. Then $2m-1 \leq v_{ss'} + v_{sh'} + v_{hh'}$.

Proof. Each napkin covers exactly one cell of type (s, s'). Removing all napkins covering a vicious cell of this type, we get another collection of napkins, which covers each cell of type (s, s') either 0 or M times depending on whether the cell is vicious or not. Hence $(m^2 - v_{ss'})M$ napkins are left and throughout the proof of Claim 2 we will consider only these remaining napkins. Now, using a red pen, write in each cell the number of napkins covering it. Notice that a cell containing a red number greater than M is surely vicious.

We call two cells *neighbors* if they can be simultaneously covered by some napkin. So, each cell of type (h, h') has not more than four neighbors of type (s, s'), while each cell of type (s, h') has not more than two neighbors of each of the types (s, s') and (h, h'). Therefore, each red number at a cell of type (h, h') does not exceed $4M$, while each red number at a cell of type (s, h') does not exceed $2M$.

Let x, y, and z be the numbers of cells of type (h, h') whose red number belongs to $(M, 2M]$, $(2M, 3M]$, and $(3M, 4M]$, respectively. All these cells are vicious, hence $x+y+z \leq v_{hh'}$. The red numbers appearing in cells of type (h, h') clearly sum up to $(m^2 - v_{ss'})M$. Bounding each of these numbers by a multiple of M we get

$$(m^2 - v_{ss'})M \leq \big((m-1)^2 - (x+y+z)\big)\, M + 2xM + 3yM + 4zM,$$

i.e.

$$2m-1 \leq v_{ss'} + x + 2y + 3z \leq v_{ss'} + v_{hh'} + y + 2z.$$

So, to prove the claim it suffices to prove that $y + 2z \leq v_{sh'}$.

For a cell δ of type (h, h') and a cell β of type (s, h') we say that δ *forces* β if there are more than M napkins covering both of them. Since each red number in a cell of type (s, h') does not exceed $2M$, it cannot be forced by more than one cell.

On the other hand, if a red number in a (h, h')-cell belongs to $(2M, 3M]$, then it forces at least one of its neighbors of type (s, h') (since the sum of red numbers in their cells is greater than $2M$). Analogously, an (h, h')-cell with the red number in $(3M, 4M]$ forces both its neighbors of type (s, h'), since their red numbers do not exceed $2M$. Therefore there are at least $y + 2z$ forced cells and clearly all of them are vicious, as desired. □

Claim 3. We have

$$2m-1 \leq \frac{a}{S^2} + \frac{b+c}{2SH} + \frac{d}{H^2}. \tag{2}$$

Proof. Averaging the previous result over all S^2H^2 possibilities for the quadruple (s, s', h, h'), we get $2m - 1 \leq \frac{a}{S^2} + \frac{b}{SH} + \frac{d}{H^2}$. Due to the symmetry between rows and columns, the same estimate holds with b replaced by c. Averaging these two inequalities we arrive at our claim. □

Now let us multiply (2) by H^2, multiply (1) by $(2SH - H^2)$ and add them; we get

$$H^2(2m-1)+(2SH-H^2)m \leq a\cdot\frac{H^2+2SH-H^2}{S^2}+(b+c)\frac{H^2+2SH-H^2}{2SH}+d = a\cdot\frac{2H}{S}+b+c+d.$$

The left-hand side is exactly $H^2(m-1) + 2SHm$, while the right-hand side does not exceed $a + b + c + d$ since $2H \leq S$. Hence we come to the desired inequality.

Comment 1. Claim 2 is the key difference between the two solutions, because it allows to get rid of the notions of rich and poor cells. However, one may prove it by the "strawberry method" as well. It suffices to put a strawberry on each cell which is bad for an s-row, and a strawberry on each cell which is bad for an h'-column. Then each cell would contain not more than one strawberry.

Comment 2. Both solutions above work if the residue of the table size T modulo the napkin size R is at least $\frac{2}{3}R$, or equivalently if $T = Sm + H(m-1)$ and $R = S + H$ for some positive integers S, H, m such that $S \geq 2H$. Here we discuss all other possible combinations.

Case 1. If $2H \geq S \geq H/2$, then the sharp bound for the number of vicious cells is $mS^2 + (m-1)H^2$; it can be obtained by the same methods as in any of the solutions. To obtain an example showing that the bound is sharp, one may simply remove the napkins of the third kind from the example in Solution 1 (with an obvious change in the numbers).

Case 2. If $2S \leq H$, the situation is more difficult. If $(S + H)^2 > 2H^2$, then the answer and the example are the same as in the previous case; otherwise the answer is $(2m-1)S^2 + 2SH(m-1)$, and the example is provided simply by $(m-1)^2$ nonintersecting napkins.

Now we sketch the proof of both estimates for Case 2. We introduce a more appropriate notation based on that from Solution 2. Denote by a_- and a_+ the number of cells of class A that contain the number which is strictly less than M and strictly greater than M, respectively. The numbers $b_\pm$, $c_\pm$, and $d_\pm$ are defined in a similar way. One may notice that the proofs of Claim 1 and Claims 2, 3 lead in fact to the inequalities

$$m-1 \leq \frac{b_- + c_-}{2SH} + \frac{d_+}{H^2} \quad \text{and} \quad 2m-1 \leq \frac{a}{S^2} + \frac{b_+ + c_+}{2SH} + \frac{d_+}{H^2}$$

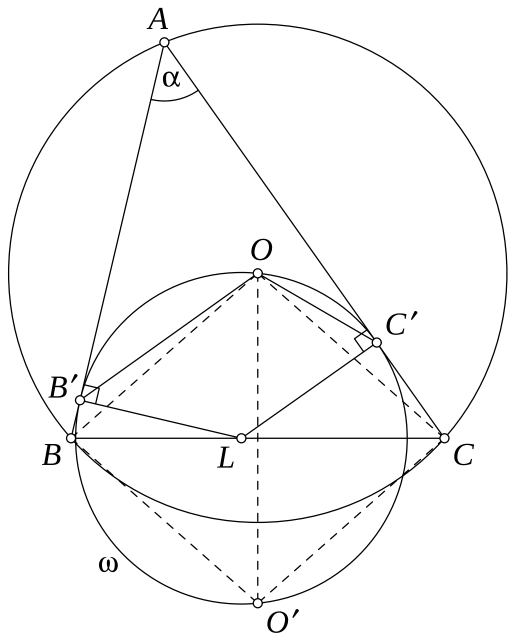
(to obtain the first one, one needs to look at the big lines instead of the small ones). Combining these inequalities, one may obtain the desired estimates.

These estimates can also be proved in some different ways, e.g. without distinguishing rich and poor cells.

G1

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to AB at B' and to AC at C'. Suppose also that the circumcenter O of the triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Solution. The point B', being the perpendicular foot of L, is an interior point of side AB. Analogously, C' lies in the interior of AC. The point O is located inside the triangle $AB'C'$, hence $\angle COB < \angle C'OB'$.



Let $\alpha = \angle CAB$. The angles $\angle CAB$ and $\angle C'OB'$ are inscribed into the two circles with centers O and L, respectively, so $\angle COB = 2\angle CAB = 2\alpha$ and $2\angle C'OB' = 360° - \angle C'LB'$. From the kite $AB'LC'$ we have $\angle C'LB' = 180° - \angle C'AB' = 180° - \alpha$. Combining these, we get

$$2\alpha = \angle COB < \angle C'OB' = \frac{360° - \angle C'LB'}{2} = \frac{360° - (180° - \alpha)}{2} = 90° + \frac{\alpha}{2},$$

so

$$\alpha < 60°.$$

Let O' be the reflection of O in the line BC. In the quadrilateral $ABO'C$ we have

$$\angle CO'B + \angle CAB = \angle COB + \angle CAB = 2\alpha + \alpha < 180°,$$

so the point O' is outside the circle ABC. Hence, O and O' are two points of ω such that one of them lies inside the circumcircle, while the other one is located outside. Therefore, the two circles intersect.

Comment. There are different ways of reducing the statement of the problem to the case $\alpha < 60^\circ$. E.g., since the point O lies in the interior of the isosceles triangle $AB'C'$, we have $OA < AB'$. So, if $AB' \leq 2LB'$ then $OA < 2LO$, which means that ω intersects the circumcircle of ABC. Hence the only interesting case is $AB' > 2LB'$, and this condition implies $\angle CAB = 2\angle B'AL < 2 \cdot 30^\circ = 60^\circ$.

G2

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Solution 1. Let M be the point of intersection of the diagonals A_1A_3 and A_2A_4. On each diagonal choose a direction and let x, y, z, and w be the signed distances from M to the points A_1, A_2, A_3, and A_4, respectively.

Let ω_1 be the circumcircle of the triangle $A_2A_3A_4$ and let B_1 be the second intersection point of ω_1 and A_1A_3 (thus, $B_1 = A_3$ if and only if A_1A_3 is tangent to ω_1). Since the expression $O_1A_1^2 - r_1^2$ is the power of the point A_1 with respect to ω_1, we get

$$O_1A_1^2 - r_1^2 = A_1B_1 \cdot A_1A_3.$$

On the other hand, from the equality $MB_1 \cdot MA_3 = MA_2 \cdot MA_4$ we obtain $MB_1 = yw/z$. Hence, we have

$$O_1A_1^2 - r_1^2 = \left(\frac{yw}{z} - x\right)(z - x) = \frac{z - x}{z}(yw - xz).$$

Substituting the analogous expressions into the sought sum we get

$$\sum_{i=1}^{4} \frac{1}{O_iA_i^2 - r_i^2} = \frac{1}{yw - xz}\left(\frac{z}{z - x} - \frac{w}{w - y} + \frac{x}{x - z} - \frac{y}{y - w}\right) = 0,$$

as desired.

Comment. One might reformulate the problem by assuming that the quadrilateral $A_1A_2A_3A_4$ is convex. This should not really change the difficulty, but proofs that distinguish several cases may become shorter.

Solution 2. Introduce a Cartesian coordinate system in the plane. Every circle has an equation of the form $p(x, y) = x^2 + y^2 + l(x, y) = 0$, where $l(x, y)$ is a polynomial of degree at most 1. For any point $A = (x_A, y_A)$ we have $p(x_A, y_A) = d^2 - r^2$, where d is the distance from A to the center of the circle and r is the radius of the circle.

For each i in $\{1, 2, 3, 4\}$ let $p_i(x, y) = x^2 + y^2 + l_i(x, y) = 0$ be the equation of the circle with center O_i and radius r_i and let d_i be the distance from A_i to O_i. Consider the equation

$$\sum_{i=1}^{4} \frac{p_i(x, y)}{d_i^2 - r_i^2} = 1. \tag{1}$$

Since the coordinates of the points A_1, A_2, A_3, and A_4 satisfy (1) but these four points do not lie on a circle or on an line, equation (1) defines neither a circle, nor a line. Hence, the equation is an identity and the coefficient of the quadratic term $x^2 + y^2$ also has to be zero, i.e.

$$\sum_{i=1}^{4} \frac{1}{d_i^2 - r_i^2} = 0.$$

Comment. Using the determinant form of the equation of the circle through three given points, the same solution can be formulated as follows.

For $i = 1, 2, 3, 4$ let (u_i, v_i) be the coordinates of A_i and define

$$\Delta = \begin{vmatrix} u_1^2+v_1^2 & u_1 & v_1 & 1 \\ u_2^2+v_2^2 & u_2 & v_2 & 1 \\ u_3^2+v_3^2 & u_3 & v_3 & 1 \\ u_4^2+v_4^2 & u_4 & v_4 & 1 \end{vmatrix} \quad \text{and} \quad \Delta_i = \begin{vmatrix} u_{i+1} & v_{i+1} & 1 \\ u_{i+2} & v_{i+2} & 1 \\ u_{i+3} & v_{i+3} & 1 \end{vmatrix},$$

where $i+1$, $i+2$, and $i+3$ have to be read modulo 4 as integers in the set $\{1, 2, 3, 4\}$.

Expanding $\begin{vmatrix} u_1 & v_1 & 1 & 1 \\ u_2 & v_2 & 1 & 1 \\ u_3 & v_3 & 1 & 1 \\ u_4 & v_4 & 1 & 1 \end{vmatrix} = 0$ along the third column, we get $\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 = 0$.

The circle through A_{i+1}, A_{i+2}, and A_{i+3} is given by the equation

$$\frac{1}{\Delta_i} \begin{vmatrix} x^2+y^2 & x & y & 1 \\ u_{i+1}^2+v_{i+1}^2 & u_{i+1} & v_{i+1} & 1 \\ u_{i+2}^2+v_{i+2}^2 & u_{i+2} & v_{i+2} & 1 \\ u_{i+3}^2+v_{i+3}^2 & u_{i+3} & v_{i+3} & 1 \end{vmatrix} = 0. \tag{2}$$

On the left-hand side, the coefficient of $x^2 + y^2$ is equal to 1. Substituting (u_i, v_i) for (x, y) in (2) we obtain the power of point A_i with respect to the circle through A_{i+1}, A_{i+2}, and A_{i+3}:

$$d_i^2 - r_i^2 = \frac{1}{\Delta_i} \begin{vmatrix} u_i^2+v_i^2 & u_i & v_i & 1 \\ u_{i+1}^2+v_{i+1}^2 & u_{i+1} & v_{i+1} & 1 \\ u_{i+2}^2+v_{i+2}^2 & u_{i+2} & v_{i+2} & 1 \\ u_{i+3}^2+v_{i+3}^2 & u_{i+3} & v_{i+3} & 1 \end{vmatrix} = (-1)^{i+1} \frac{\Delta}{\Delta_i}.$$

Thus, we have

$$\sum_{i=1}^{4} \frac{1}{d_i^2 - r_i^2} = \frac{\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4}{\Delta} = 0.$$

G3

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC, and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB. Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

Solution. Denote by P, Q, R, and S the projections of E on the lines DA, AB, BC, and CD respectively. The points P and Q lie on the circle with diameter AE, so $\angle QPE = \angle QAE$; analogously, $\angle QRE = \angle QBE$. So $\angle QPE + \angle QRE = \angle QAE + \angle QBE = 90^\circ$. By similar reasons, we have $\angle SPE + \angle SRE = 90^\circ$, hence we get $\angle QPS + \angle QRS = 90^\circ + 90^\circ = 180^\circ$, and the quadrilateral $PQRS$ is inscribed in ω_E. Analogously, all four projections of F onto the sides of $ABCD$ lie on ω_F.

Denote by K the meeting point of the lines AD and BC. Due to the arguments above, there is no loss of generality in assuming that A lies on segment DK. Suppose that $\angle CKD \geq 90^\circ$; then the circle with diameter CD covers the whole quadrilateral $ABCD$, so the points E, F cannot lie inside this quadrilateral. Hence our assumption is wrong. Therefore, the lines EP and BC intersect at some point P', while the lines ER and AD intersect at some point R'.

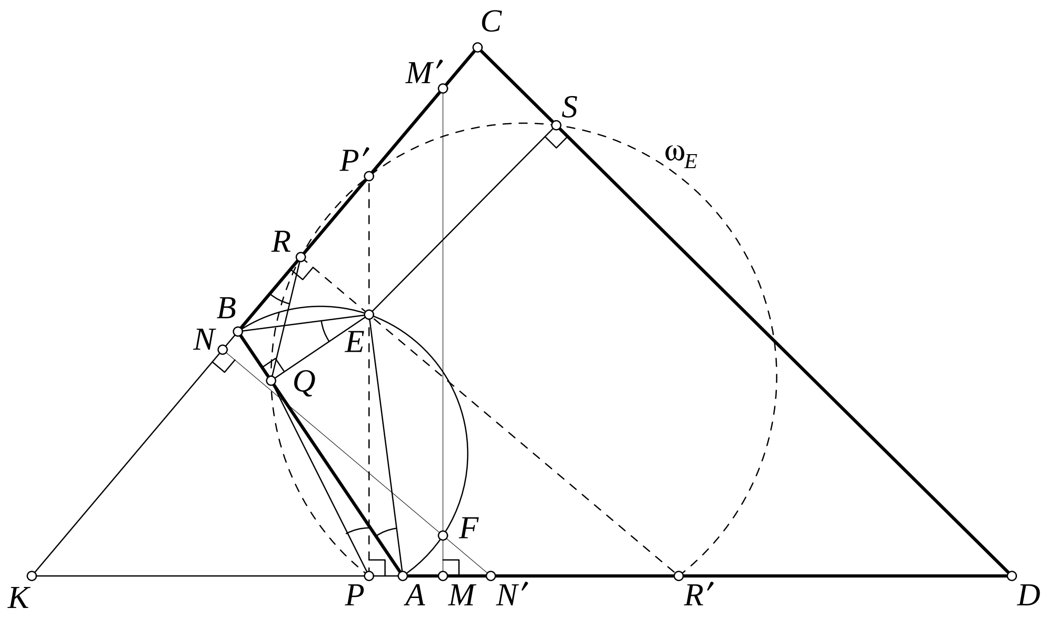


Figure 1

We claim that the points P' and R' also belong to ω_E. Since the points R, E, Q, B are concyclic, $\angle QRK = \angle QEB = 90^\circ - \angle QBE = \angle QAE = \angle QPE$. So $\angle QRK = \angle QPP'$, which means that the point P' lies on ω_E. Analogously, R' also lies on ω_E.

In the same manner, denote by M and N the projections of F on the lines AD and BC

respectively, and let $M' = FM \cap BC$, $N' = FN \cap AD$. By the same arguments, we obtain that the points M' and N' belong to ω_F.

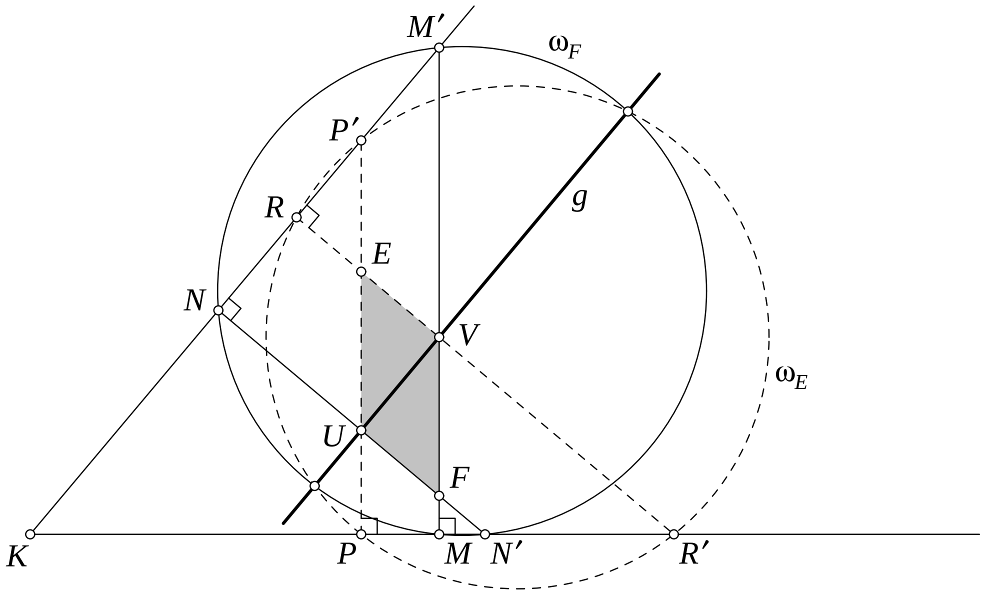


Figure 2

Now we concentrate on Figure 2, where all unnecessary details are removed. Let $U = NN' \cap PP'$, $V = MM' \cap RR'$. Due to the right angles at N and P, the points N, N', P, P' are concyclic, so $UN \cdot UN' = UP \cdot UP'$ which means that U belongs to the radical axis g of the circles ω_E and ω_F. Analogously, V also belongs to g.

Finally, since $EUFV$ is a parallelogram, the radical axis UV of ω_E and ω_F bisects EF.

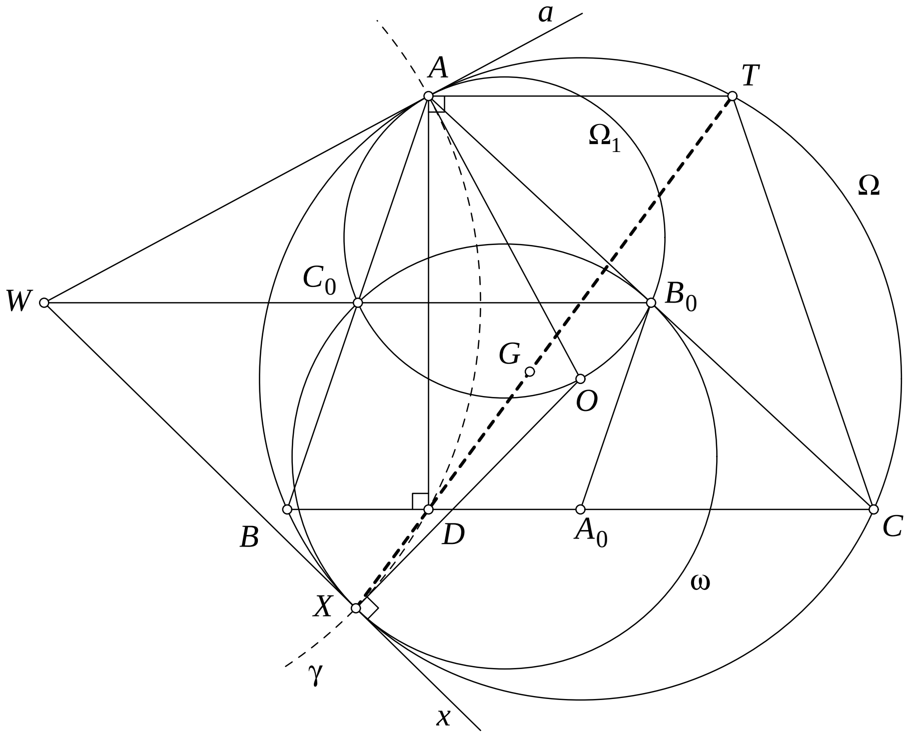
G4

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.

Solution 1. If $AB = AC$, then the statement is trivial. So without loss of generality we may assume $AB < AC$. Denote the tangents to Ω at points A and X by a and x, respectively.

Let Ω_1 be the circumcircle of triangle AB_0C_0. The circles Ω and Ω_1 are homothetic with center A, so they are tangent at A, and a is their radical axis. Now, the lines a, x, and B_0C_0 are the three radical axes of the circles Ω, Ω_1, and ω. Since $a \nparallel B_0C_0$, these three lines are concurrent at some point W.

The points A and D are symmetric with respect to the line B_0C_0; hence $WX = WA = WD$. This means that W is the center of the circumcircle γ of triangle ADX. Moreover, we have $\angle WAO = \angle WXO = 90°$, where O denotes the center of Ω. Hence $\angle AWX + \angle AOX = 180°$.



Denote by T the second intersection point of Ω and the line DX. Note that O belongs to Ω_1. Using the circles γ and Ω, we find $\angle DAT = \angle ADX - \angle ATD = \frac{1}{2}(360° - \angle AWX) - \frac{1}{2}\angle AOX = 180° - \frac{1}{2}(\angle AWX + \angle AOX) = 90°$. So, $AD \perp AT$, and hence $AT \parallel BC$. Thus, $ATCB$ is an isosceles trapezoid inscribed in Ω.

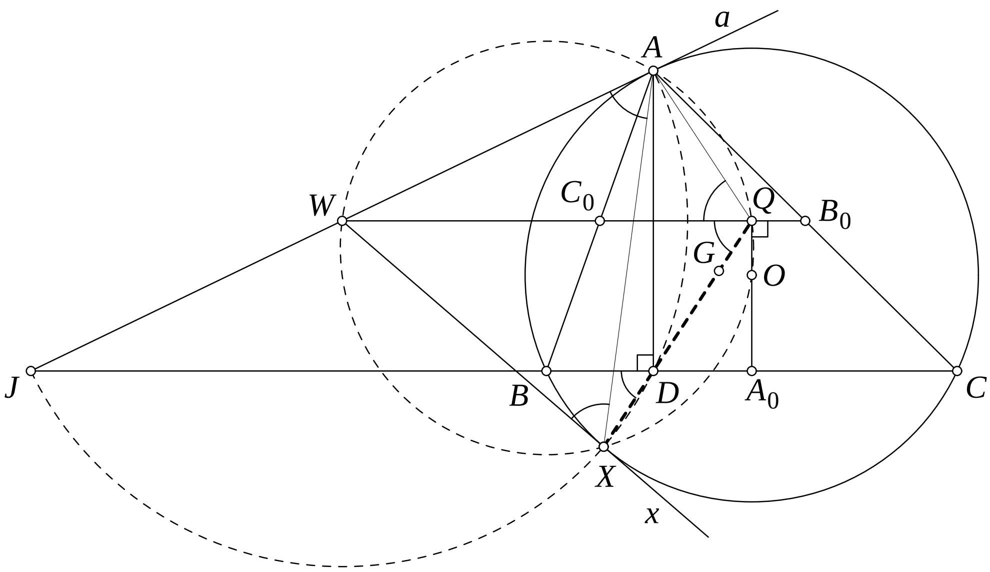
Denote by A_0 the midpoint of BC, and consider the image of $ATCB$ under the homothety h with center G and factor $-\frac{1}{2}$. We have $h(A) = A_0$, $h(B) = B_0$, and $h(C) = C_0$. From the symmetry about B_0C_0, we have $\angle TCB = \angle CBA = \angle B_0C_0A = \angle DC_0B_0$. Using $AT \parallel DA_0$, we conclude $h(T) = D$. Hence the points D, G, and T are collinear, and X lies on the same line.

Solution 2. We define the points A_0, O, and W as in the previous solution and we concentrate on the case $AB < AC$. Let Q be the perpendicular projection of A_0 on B_0C_0.

Since $\angle WAO = \angle WQO = \angle OXW = 90°$, the five points A, W, X, O, and Q lie on a common circle. Furthermore, the reflections with respect to B_0C_0 and OW map A to D and X, respectively. For these reasons, we have

$$\angle WQD = \angle AQW = \angle AXW = \angle WAX = \angle WQX.$$

Thus the three points Q, D, and X lie on a common line, say ℓ.



To complete the argument, we note that the homothety centered at G sending the triangle ABC to the triangle $A_0B_0C_0$ maps the altitude AD to the altitude A_0Q. Therefore it maps D to Q, so the points D, G, and Q are collinear. Hence G lies on ℓ as well.

Comment. There are various other ways to prove the collinearity of Q, D, and X obtained in the middle part of Solution 2. Introduce for instance the point J where the lines AW and BC intersect. Then the four points A, D, X, and J lie at the same distance from W, so the quadrilateral $ADXJ$ is cyclic. In combination with the fact that $AWXQ$ is cyclic as well, this implies

$$\angle JDX = \angle JAX = \angle WAX = \angle WQX.$$

Since $BC \parallel WQ$, it follows that Q, D, and X are indeed collinear.

G5

Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD, and KP are either parallel or concurrent.

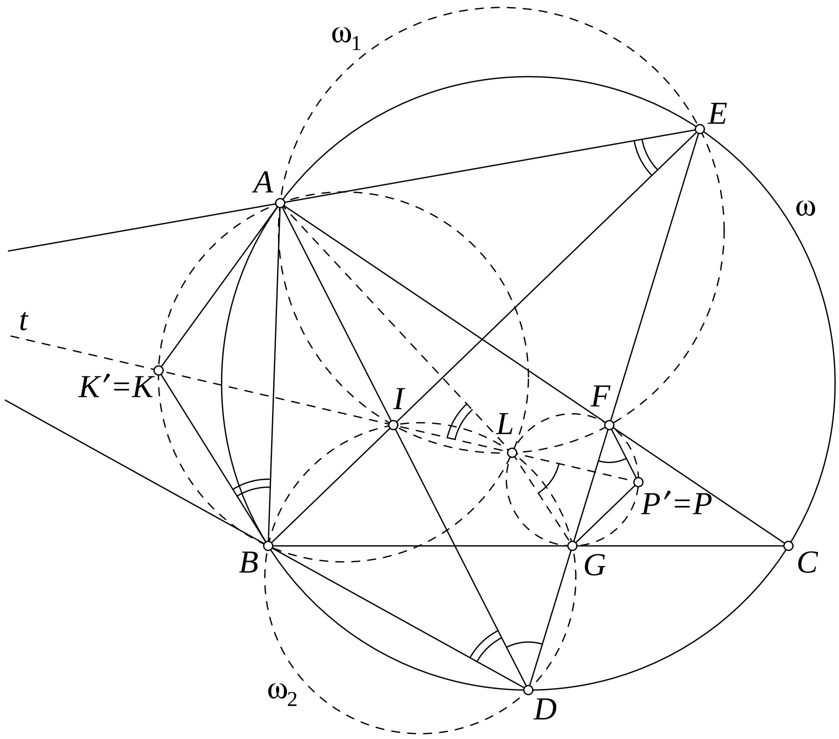
Solution 1. Since

$$\angle IAF = \angle DAC = \angle BAD = \angle BED = \angle IEF$$

the quadrilateral $AIFE$ is cyclic. Denote its circumcircle by ω_1. Similarly, the quadrilateral $BDGI$ is cyclic; denote its circumcircle by ω_2.

The line AE is the radical axis of ω and ω_1, and the line BD is the radical axis of ω and ω_2. Let t be the radical axis of ω_1 and ω_2. These three lines meet at the radical center of the three circles, or they are parallel to each other. We will show that t is in fact the line PK.

Let L be the second intersection point of ω_1 and ω_2, so $t = IL$. (If the two circles are tangent to each other then $L = I$ and t is the common tangent.)



Let the line t meet the circumcircles of the triangles ABL and FGL at $K' \neq L$ and $P' \neq L$, respectively. Using oriented angles we have

$$\angle(AB, BK') = \angle(AL, LK') = \angle(AL, LI) = \angle(AE, EI) = \angle(AE, EB) = \angle(AB, BK),$$

so $BK' \parallel BK$. Similarly we have $AK' \parallel AK$, and therefore $K' = K$. Next, we have

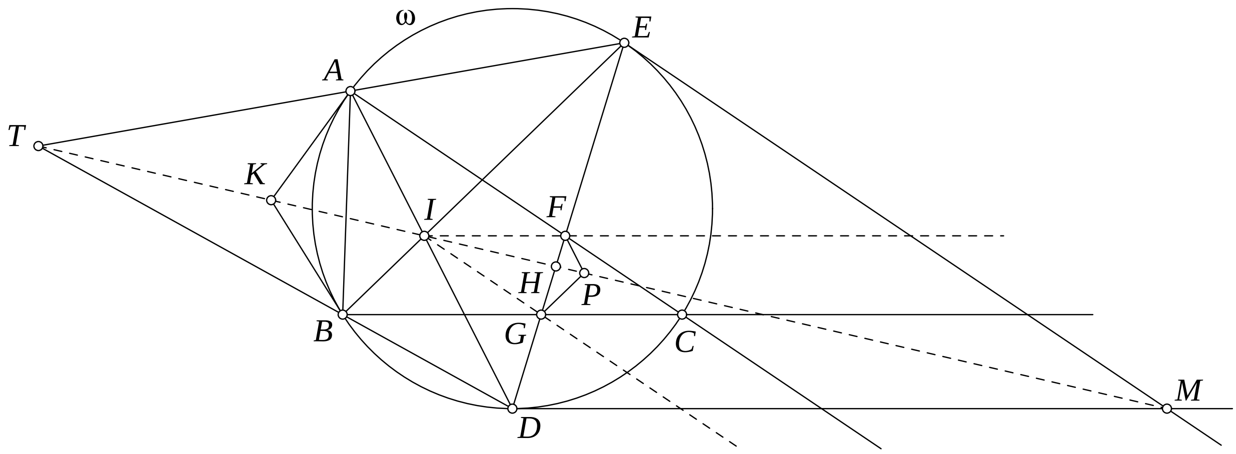
$$\angle(P'F, FG) = \angle(P'L, LG) = \angle(IL, LG) = \angle(ID, DG) = \angle(AD, DE) = \angle(PF, FG),$$

hence $P'F \parallel PF$ and similarly $P'G \parallel PG$. Therefore $P' = P$. This means that t passes through K and P, which finishes the proof.

Solution 2. Let M be the intersection point of the tangents to ω at D and E, and let the lines AE and BD meet at T; if AE and BD are parallel, then let T be their common ideal point. It is well-known that the points K and M lie on the line TI (as a consequence of Pascal's theorem, applied to the inscribed degenerate hexagons $AADBBE$ and $ADDBEE$).

The lines AD and BE are the angle bisectors of the angles $\angle CAB$ and $\angle ABC$, respectively, so D and E are the midpoints of the arcs BC and CA of the circle ω, respectively. Hence, DM is parallel to BC and EM is parallel to AC.

Apply Pascal's theorem to the degenerate hexagon $CADDEB$. By the theorem, the points $CA \cap DE = F$, $AD \cap EB = I$ and the common ideal point of lines DM and BC are collinear, therefore FI is parallel to BC and DM. Analogously, the line GI is parallel to AC and EM.



Now consider the homothety with scale factor $-\frac{FG}{ED}$ which takes E to G and D to F. Since the triangles EDM and GFI have parallel sides, the homothety takes M to I. Similarly, since the triangles DEI and FGP have parallel sides, the homothety takes I to P. Hence, the points M, I, P and the homothety center H must lie on the same line. Therefore, the point P also lies on the line $TKIM$.

Comment. One may prove that $IF \parallel BC$ and $IG \parallel AC$ in a more elementary way. Since $\angle ADE = \angle EDC$ and $\angle DEB = \angle CED$, the points I and C are symmetric about DE. Moreover, since the arcs AE and EC are equal and the arcs CD and DB are equal, we have $\angle CFG = \angle FGC$, so the triangle CFG is isosceles. Hence, the quadrilateral $IFCG$ is a rhombus.

G6

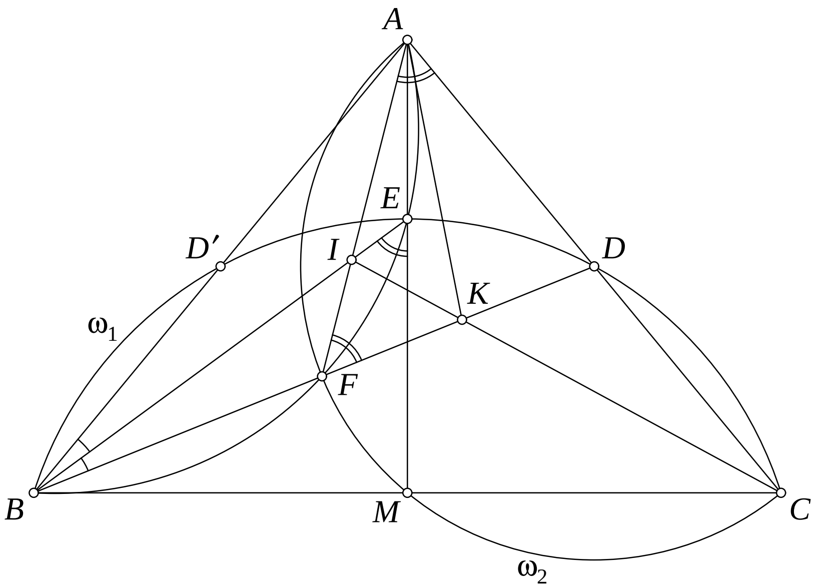
Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

Solution 1. Let D' be the midpoint of the segment AB, and let M be the midpoint of BC. By symmetry at line AM, the point D' has to lie on the circle BCD. Since the arcs $D'E$ and ED of that circle are equal, we have $\angle ABI = \angle D'BE = \angle EBD = IBK$, so I lies on the angle bisector of $\angle ABK$. For this reason it suffices to prove in the sequel that the ray AI bisects the angle $\angle BAK$.

From

$$\angle DFA = 180^\circ - \angle BFA = 180^\circ - \angle BEA = \angle MEB = \frac{1}{2}\angle CEB = \frac{1}{2}\angle CDB$$

we derive $\angle DFA = \angle DAF$ so the triangle AFD is isosceles with $AD = DF$.



Applying Menelaus's theorem to the triangle ADF with respect to the line CIK, and applying the angle bisector theorem to the triangle ABF, we infer

$$1 = \frac{AC}{CD} \cdot \frac{DK}{KF} \cdot \frac{FI}{IA} = 2 \cdot \frac{DK}{KF} \cdot \frac{BF}{AB} = 2 \cdot \frac{DK}{KF} \cdot \frac{BF}{2 \cdot AD} = \frac{DK}{KF} \cdot \frac{BF}{AD}$$

and therefore

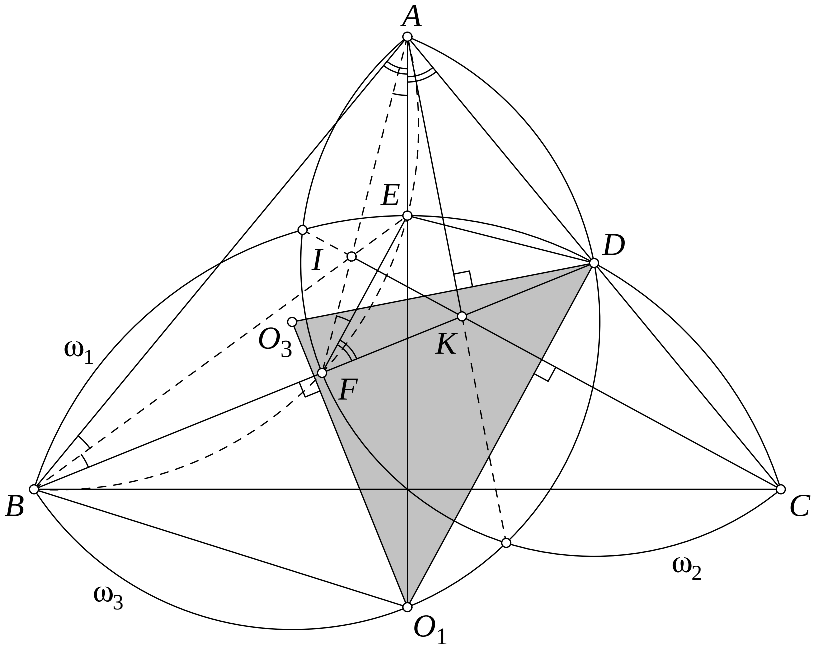
$$\frac{BD}{AD} = \frac{BF + FD}{AD} = \frac{BF}{AD} + 1 = \frac{KF}{DK} + 1 = \frac{DF}{DK} = \frac{AD}{DK}.$$

It follows that the triangles ADK and BDA are similar, hence $\angle DAK = \angle ABD$. Then

$$\angle IAB = \angle AFD - \angle ABD = \angle DAF - \angle DAK = \angle KAI$$

shows that the point K is indeed lying on the angle bisector of $\angle BAK$.

Solution 2. It can be shown in the same way as in the first solution that I lies on the angle bisector of $\angle ABK$. Here we restrict ourselves to proving that KI bisects $\angle AKB$.



Denote the circumcircle of triangle BCD and its center by ω_1 and by O_1, respectively. Since the quadrilateral $ABFE$ is cyclic, we have $\angle DFE = \angle BAE = \angle DAE$. By the same reason, we have $\angle EAF = \angle EBF = \angle ABE = \angle AFE$. Therefore $\angle DAF = \angle DFA$, and hence $DF = DA = DC$. So triangle AFC is inscribed in a circle ω_2 with center D.

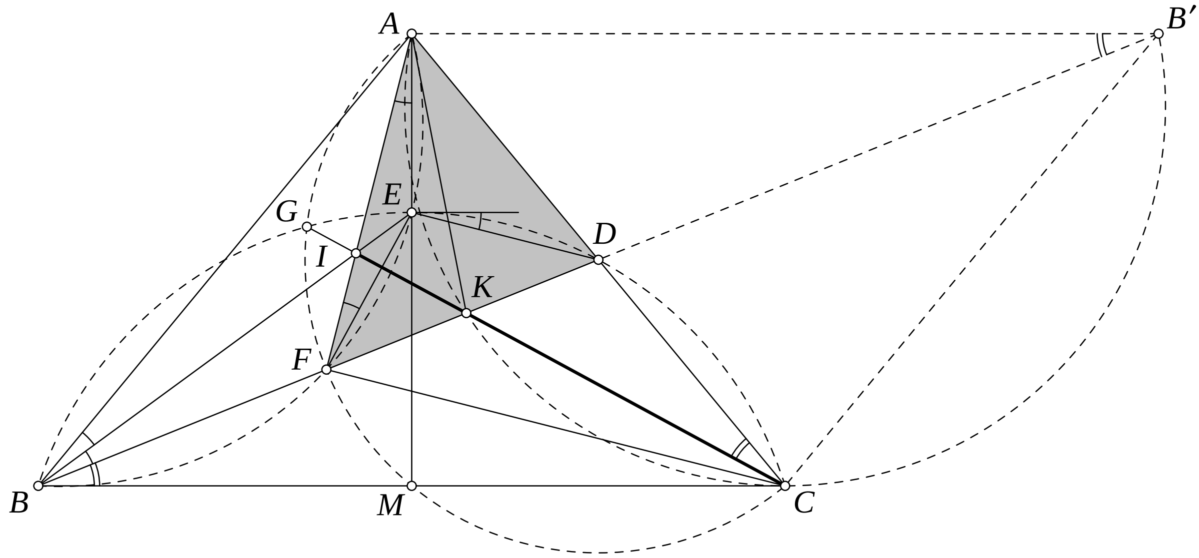
Denote the circumcircle of triangle ABD by ω_3, and let its center be O_3. Since the arcs BE and EC of circle ω_1 are equal, and the triangles ADE and FDE are congruent, we have $\angle AO_1B = 2\angle BDE = \angle BDA$, so O_1 lies on ω_3. Hence $\angle O_3O_1D = \angle O_3DO_1$.

The line BD is the radical axis of ω_1 and ω_3. Point C belongs to the radical axis of ω_1 and ω_2, and I also belongs to it since $AI \cdot IF = BI \cdot IE$. Hence $K = BD \cap CI$ is the radical center of ω_1, ω_2, and ω_3, and AK is the radical axis of ω_2 and ω_3. Now, the radical axes AK, BK and IK are perpendicular to the central lines O_3D, O_3O_1 and O_1D, respectively. By $\angle O_3O_1D = \angle O_3DO_1$, we get that KI is the angle bisector of $\angle AKB$.

Solution 3. Again, let M be the midpoint of BC. As in the previous solutions, we can deduce $\angle ABI = \angle IBK$. We show that the point I lies on the angle bisector of $\angle KAB$.

Let G be the intersection point of the circles AFC and BCD, different from C. The lines

CG, AF, and BE are the radical axes of the three circles $AGFC$, CDB, and $ABFE$, so $I = AF \cap BE$ is the radical center of the three circles and CG also passes through I.



The angle between line DE and the tangent to the circle BCD at E is equal to $\angle EBD = \angle EAF = \angle ABE = \angle AFE$. As the tangent at E is perpendicular to AM, the line DE is perpendicular to AF. The triangle AFE is isosceles, so DE is the perpendicular bisector of AF and thus $AD = DF$. Hence, the point D is the center of the circle AFC, and this circle passes through M as well since $\angle AMC = 90°$.

Let B' be the reflection of B in the point D, so $ABCB'$ is a parallelogram. Since $DC = DG$ we have $\angle GCD = \angle DBC = \angle KB'A$. Hence, the quadrilateral $AKCB'$ is cyclic and thus $\angle CAK = \angle CB'K = \angle ABD = 2\angle MAI$. Then

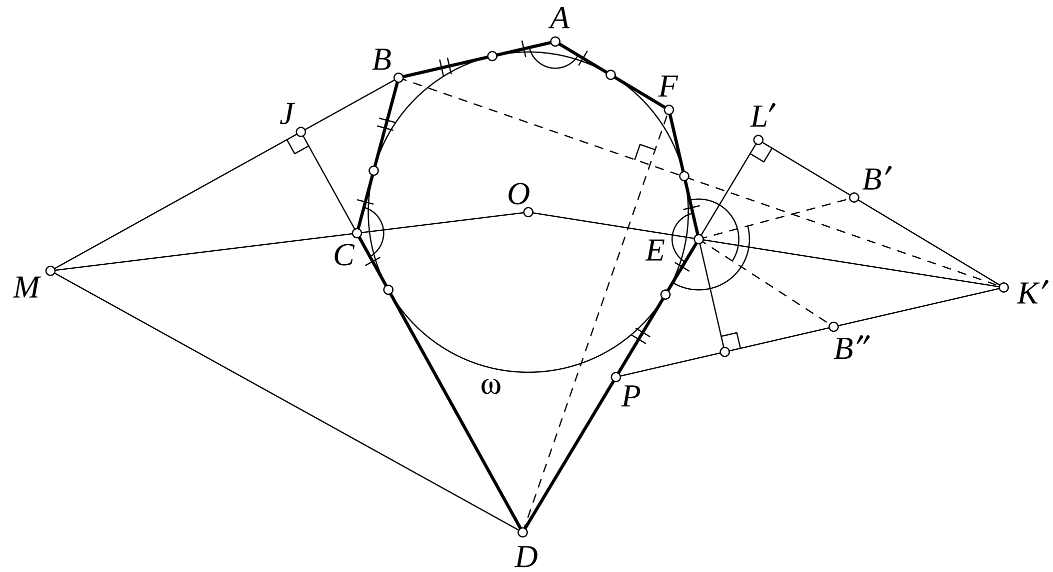
$$\angle IAB = \angle MAB - \angle MAI = \frac{1}{2}\angle CAB - \frac{1}{2}\angle CAK = \frac{1}{2}\angle KAB$$

and therefore AI is the angle bisector of $\angle KAB$.

G7

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Solution 1. Since ω and the circumcircle of triangle ACE are concentric, the tangents from A, C, and E to ω have equal lengths; that means that $AB = BC$, $CD = DE$, and $EF = FA$. Moreover, we have $\angle BCD = \angle DEF = \angle FAB$.



Consider the rotation around point D mapping C to E; let B' and L' be the images of the points B and J, respectively, under this rotation. Then one has $DJ = DL'$ and $B'L' \perp DE$; moreover, the triangles $B'ED$ and BCD are congruent. Since $\angle DEO < 90°$, the lines EO and $B'L'$ intersect at some point K'. We intend to prove that $K'B \perp DF$; this would imply $K = K'$, therefore $L = L'$, which proves the problem statement.

Analogously, consider the rotation around F mapping A to E; let B'' be the image of B under this rotation. Then the triangles FAB and FEB'' are congruent. We have $EB'' = AB = BC = EB'$ and $\angle FEB'' = \angle FAB = \angle BCD = \angle DEB'$, so the points B' and B'' are symmetrical with respect to the angle bisector EO of $\angle DEF$. So, from $K'B' \perp DE$ we get $K'B'' \perp EF$.

From these two relations we obtain

$$K'D^2 - K'E^2 = B'D^2 - B'E^2 \quad \text{and} \quad K'E^2 - K'F^2 = B''E^2 - B''F^2.$$

Adding these equalities and taking into account that $B'E = B''E$ we obtain

$$K'D^2 - K'F^2 = B'D^2 - B''F^2 = BD^2 - BF^2,$$

which means exactly that $K'B \perp DF$.

Comment. There are several variations of this solution. For instance, let us consider the intersection point M of the lines BJ and OC. Define the point K' as the reflection of M in the line DO. Then one has

$$DK'^2 - DB^2 = DM^2 - DB^2 = CM^2 - CB^2.$$

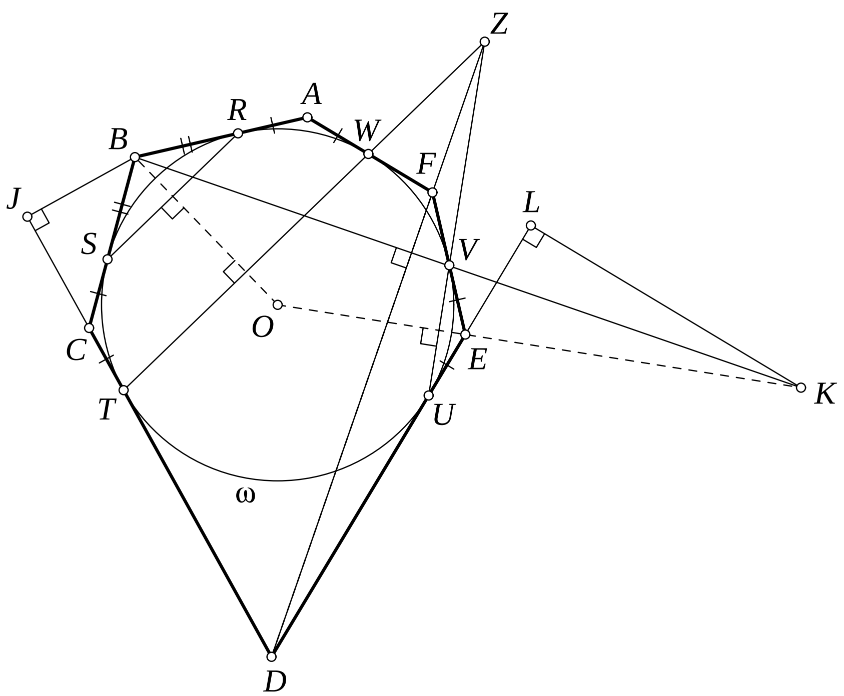
Next, consider the rotation around O which maps CM to EK'. Let P be the image of B under this rotation; so P lies on ED. Then $EF \perp K'P$, so

$$CM^2 - CB^2 = EK'^2 - EP^2 = FK'^2 - FP^2 = FK'^2 - FB^2,$$

since the triangles FEP and FAB are congruent.

Solution 2. Let us denote the points of tangency of AB, BC, CD, DE, EF, and FA to ω by R, S, T, U, V, and W, respectively. As in the previous solution, we mention that $AR = AW = CS = CT = EU = EV$.

The reflection in the line BO maps R to S, therefore A to C and thus also W to T. Hence, both lines RS and WT are perpendicular to OB, therefore they are parallel. On the other hand, the lines UV and WT are not parallel, since otherwise the hexagon $ABCDEF$ is symmetric with respect to the line BO and the lines defining the point K coincide, which contradicts the conditions of the problem. Therefore we can consider the intersection point Z of UV and WT.



Next, we recall a well-known fact that the points D, F, Z are collinear. Actually, D is the pole of the line UT, F is the pole of VW, and $Z = TW \cap UV$; so all these points belong to the polar line of $TU \cap VW$.

Now, we put O into the origin, and identify each point (say X) with the vector $\overrightarrow{OX}$. So, from now on all the products of points refer to the scalar products of the corresponding vectors.

Since $OK \perp UZ$ and $OB \perp TZ$, we have $K \cdot (Z - U) = 0 = B \cdot (Z - T)$. Next, the condition $BK \perp DZ$ can be written as $K \cdot (D - Z) = B \cdot (D - Z)$. Adding these two equalities we get

$$K \cdot (D - U) = B \cdot (D - T).$$

By symmetry, we have $D \cdot (D - U) = D \cdot (D - T)$. Subtracting this from the previous equation, we obtain $(K - D) \cdot (D - U) = (B - D) \cdot (D - T)$ and rewrite it in vector form as

$$\overrightarrow{DK} \cdot \overrightarrow{UD} = \overrightarrow{DB} \cdot \overrightarrow{TD}.$$

Finally, projecting the vectors $\overrightarrow{DK}$ and $\overrightarrow{DB}$ onto the lines UD and TD respectively, we can rewrite this equality in terms of segment lengths as $DL \cdot UD = DJ \cdot TD$, thus $DL = DJ$.

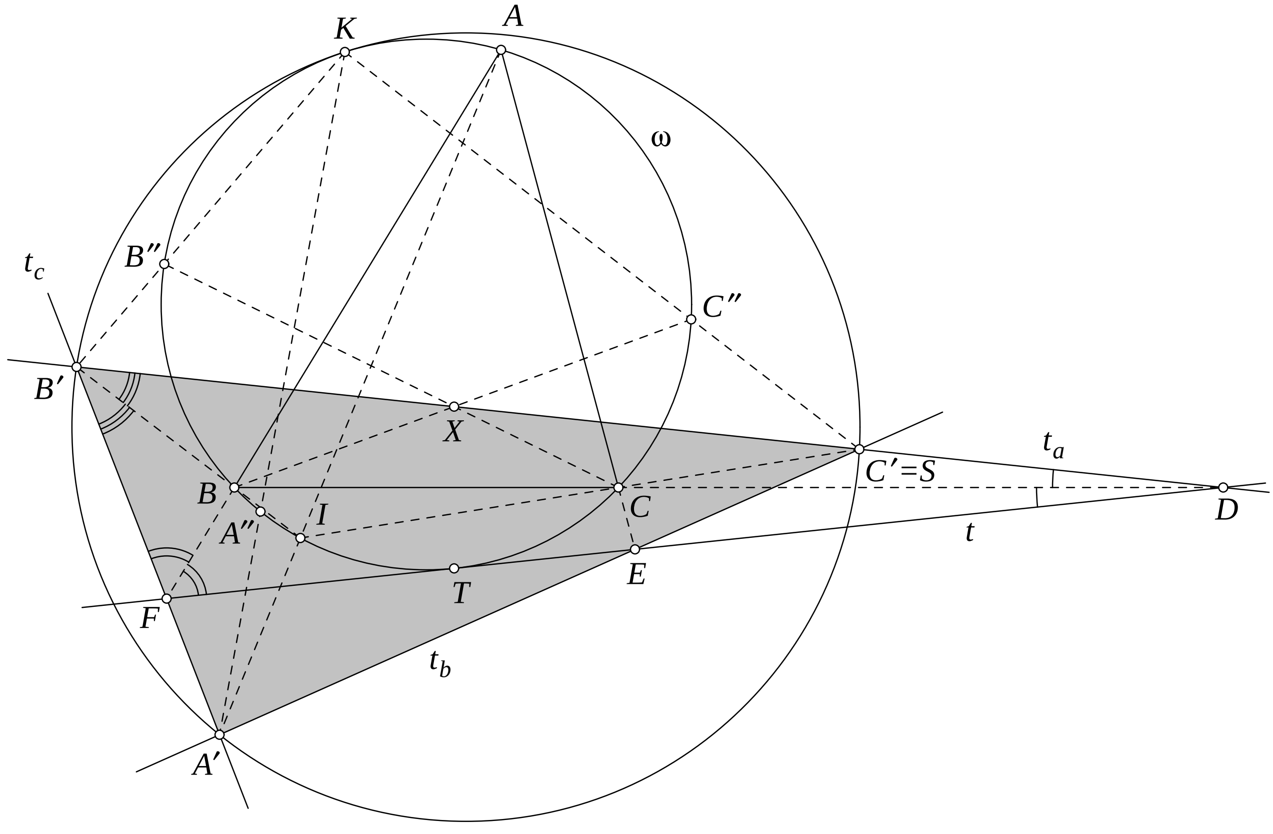
Comment. The collinearity of Z, F, and D may be shown in various more elementary ways. For instance, applying the sine theorem to the triangles DTZ and DUZ, one gets $\dfrac{\sin \angle DZT}{\sin \angle DZU} = \dfrac{\sin \angle DTZ}{\sin \angle DUZ}$; analogously, $\dfrac{\sin \angle FZW}{\sin \angle FZV} = \dfrac{\sin \angle FWZ}{\sin \angle FVZ}$. The right-hand sides are equal, hence so are the left-hand sides, which implies the collinearity of the points D, F, and Z.

There also exist purely synthetic proofs of this fact. E.g., let Q be the point of intersection of the circumcircles of the triangles ZTV and ZWU different from Z. Then QZ is the bisector of $\angle VQW$ since $\angle VQZ = \angle VTZ = \angle VUW = \angle ZQW$. Moreover, all these angles are equal to $\frac{1}{2}\angle VOW$, so $\angle VQW = \angle VOW$, hence the quadrilateral $VWOQ$ is cyclic. On the other hand, the points O, V, W lie on the circle with diameter OF due to the right angles; so Q also belongs to this circle. Since $FV = FW$, QF is also the bisector of $\angle VQW$, so F lies on QZ. Analogously, D lies on the same line.

G8

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

To avoid a large case distinction, we will use the notion of *oriented angles*. Namely, for two lines ℓ and m, we denote by $\angle(\ell, m)$ the angle by which one may rotate ℓ anticlockwise to obtain a line parallel to m. Thus, all oriented angles are considered modulo $180°$.



Solution 1. Denote by T the point of tangency of t and ω. Let $A' = t_b \cap t_c$, $B' = t_a \cap t_c$, $C' = t_a \cap t_b$. Introduce the point A'' on ω such that $TA = AA''$ ($A'' \neq T$ unless TA is a diameter). Define the points B'' and C'' in a similar way.

Since the points C and B are the midpoints of arcs TC'' and TB'', respectively, we have

$$\begin{aligned}\angle(t, B''C'') &= \angle(t, TC'') + \angle(TC'', B''C'') = 2\angle(t, TC) + 2\angle(TC'', BC'') \\ &= 2\big(\angle(t, TC) + \angle(TC, BC)\big) = 2\angle(t, BC) = \angle(t, t_a).\end{aligned}$$

It follows that t_a and $B''C''$ are parallel. Similarly, $t_b \parallel A''C''$ and $t_c \parallel A''B''$. Thus, either the triangles $A'B'C'$ and $A''B''C''$ are homothetic, or they are translates of each other. Now we will prove that they are in fact homothetic, and that the center K of the homothety belongs to ω. It would then follow that their circumcircles are also homothetic with respect to K and are therefore tangent at this point, as desired.

We need the two following claims.

Claim 1. The point of intersection X of the lines $B''C$ and BC'' lies on t_a.

Proof. Actually, the points X and T are symmetric about the line BC, since the lines CT and CB'' are symmetric about this line, as are the lines BT and BC''. □

Claim 2. The point of intersection I of the lines BB' and CC' lies on the circle ω.

Proof. We consider the case that t is not parallel to the sides of ABC; the other cases may be regarded as limit cases. Let $D = t \cap BC$, $E = t \cap AC$, and $F = t \cap AB$.

Due to symmetry, the line DB is one of the angle bisectors of the lines $B'D$ and FD; analogously, the line FB is one of the angle bisectors of the lines $B'F$ and DF. So B is either the incenter or one of the excenters of the triangle $B'DF$. In any case we have $\angle(BD, DF) + \angle(DF, FB) + \angle(B'B, B'D) = 90^\circ$, so

$$\angle(B'B, B'C') = \angle(B'B, B'D) = 90^\circ - \angle(BC, DF) - \angle(DF, BA) = 90^\circ - \angle(BC, AB).$$

Analogously, we get $\angle(C'C, B'C') = 90^\circ - \angle(BC, AC)$. Hence,

$$\angle(BI, CI) = \angle(B'B, B'C') + \angle(B'C', C'C) = \angle(BC, AC) - \angle(BC, AB) = \angle(AB, AC),$$

which means exactly that the points A, B, I, C are concyclic. □

Now we can complete the proof. Let K be the second intersection point of $B'B''$ and ω. Applying Pascal's theorem to hexagon $KB''CIBC''$ we get that the points $B' = KB'' \cap IB$ and $X = B''C \cap BC''$ are collinear with the intersection point S of CI and $C''K$. So $S = CI \cap B'X = C'$, and the points C', C'', K are collinear. Thus K is the intersection point of $B'B''$ and $C'C''$ which implies that K is the center of the homothety mapping $A'B'C'$ to $A''B''C''$, and it belongs to ω.

Solution 2. Define the points T, A', B', and C' in the same way as in the previous solution. Let X, Y, and Z be the symmetric images of T about the lines BC, CA, and AB, respectively. Note that the projections of T on these lines form a Simson line of T with respect to ABC, therefore the points X, Y, Z are also collinear. Moreover, we have $X \in B'C'$, $Y \in C'A'$, $Z \in A'B'$.

Denote $\alpha = \angle(t, TC) = \angle(BT, BC)$. Using the symmetry in the lines AC and BC, we get

$$\angle(BC, BX) = \angle(BT, BC) = \alpha \quad \text{and} \quad \angle(XC, XC') = \angle(t, TC) = \angle(YC, YC') = \alpha.$$

Since $\angle(XC, XC') = \angle(YC, YC')$, the points X, Y, C, C' lie on some circle ω_c. Define the circles ω_a and ω_b analogously. Let ω' be the circumcircle of triangle $A'B'C'$.

Now, applying MIQUEL's theorem to the four lines $A'B'$, $A'C'$, $B'C'$, and XY, we obtain that the circles ω', ω_a, ω_b, ω_c intersect at some point K. We will show that K lies on ω, and that the tangent lines to ω and ω' at this point coincide; this implies the problem statement.

Due to symmetry, we have $XB = TB = ZB$, so the point B is the midpoint of one of the arcs XZ of circle ω_b. Therefore $\angle(KB, KX) = \angle(XZ, XB)$. Analogously, $\angle(KX, KC) = \angle(XC, XY)$. Adding these equalities and using the symmetry in the line BC we get

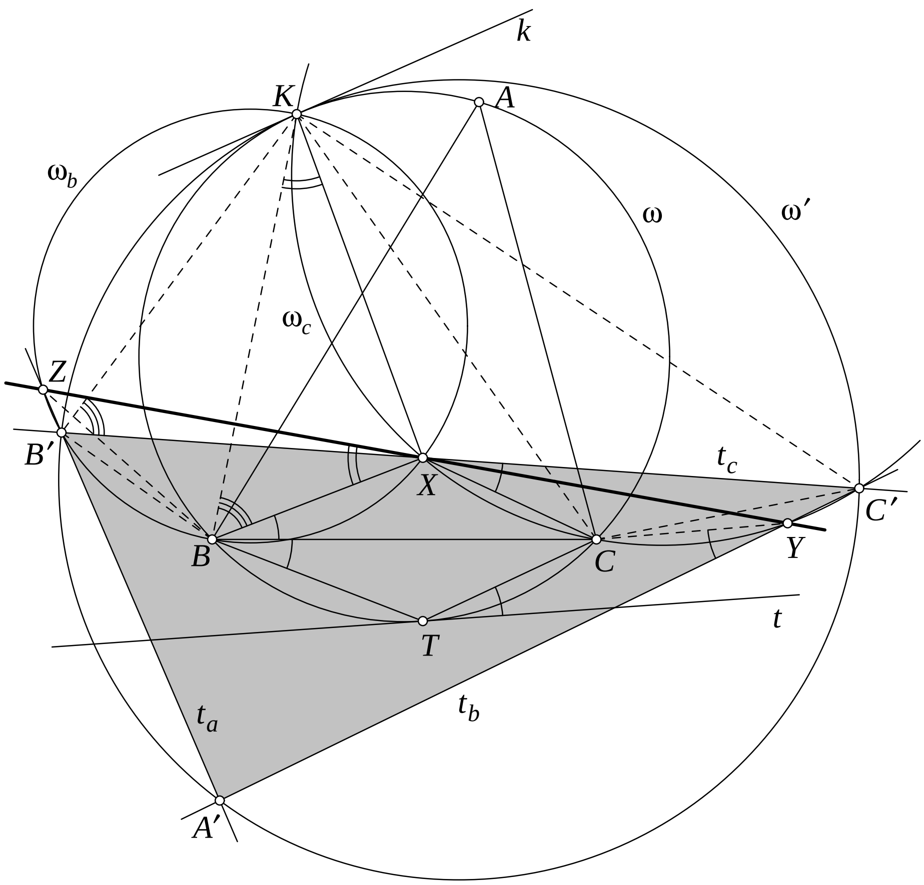
$$\angle(KB, KC) = \angle(XZ, XB) + \angle(XC, XZ) = \angle(XC, XB) = \angle(TB, TC).$$

Therefore, K lies on ω.

Next, let k be the tangent line to ω at K. We have

$$\begin{aligned}\angle(k, KC') &= \angle(k, KC) + \angle(KC, KC') = \angle(KB, BC) + \angle(XC, XC') \\ &= \big(\angle(KB, BX) - \angle(BC, BX)\big) + \alpha = \angle(KB', B'X) - \alpha + \alpha = \angle(KB', B'C'),\end{aligned}$$

which means exactly that k is tangent to ω'.



Comment. There exist various solutions combining the ideas from the two solutions presented above. For instance, one may define the point X as the reflection of T with respect to the line BC, and then introduce the point K as the second intersection point of the circumcircles of $BB'X$ and $CC'X$. Using the fact that BB' and CC' are the bisectors of $\angle(A'B', B'C')$ and $\angle(A'C', B'C')$ one can show successively that $K \in \omega$, $K \in \omega'$, and that the tangents to ω and ω' at K coincide.

N1

For any integer $d > 0$, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so for example we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f(2^k)$ divides $f(2^{k+1})$.

Solution 1. For any positive integer n, let $d(n)$ be the number of positive divisors of n. Let $n = \prod_p p^{a(p)}$ be the prime factorization of n where p ranges over the prime numbers, the integers $a(p)$ are nonnegative and all but finitely many $a(p)$ are zero. Then we have $d(n) = \prod_p (a(p)+1)$. Thus, $d(n)$ is a power of 2 if and only if for every prime p there is a nonnegative integer $b(p)$ with $a(p) = 2^{b(p)} - 1 = 1 + 2 + 2^2 + \cdots + 2^{b(p)-1}$. We then have

$$n = \prod_p \prod_{i=0}^{b(p)-1} p^{2^i}, \qquad \text{and} \quad d(n) = 2^k \quad \text{with} \quad k = \sum_p b(p).$$

Let $\mathcal{S}$ be the set of all numbers of the form p^{2^r} with p prime and r a nonnegative integer. Then we deduce that $d(n)$ is a power of 2 if and only if n is the product of the elements of some finite subset $\mathcal{T}$ of $\mathcal{S}$ that satisfies the following condition: for all $t \in \mathcal{T}$ and $s \in \mathcal{S}$ with $s \mid t$ we have $s \in \mathcal{T}$. Moreover, if $d(n) = 2^k$ then the corresponding set $\mathcal{T}$ has k elements.

Note that the set $\mathcal{T}_k$ consisting of the smallest k elements from $\mathcal{S}$ obviously satisfies the condition above. Thus, given k, the smallest n with $d(n) = 2^k$ is the product of the elements of $\mathcal{T}_k$. This n is $f(2^k)$. Since obviously $\mathcal{T}_k \subset \mathcal{T}_{k+1}$, it follows that $f(2^k) \mid f(2^{k+1})$.

Solution 2. This is an alternative to the second part of the Solution 1. Suppose k is a nonnegative integer. From the first part of Solution 1 we see that $f(2^k) = \prod_p p^{a(p)}$ with $a(p) = 2^{b(p)} - 1$ and $\sum_p b(p) = k$. We now claim that for any two distinct primes p, q with $b(q) > 0$ we have

$$m = p^{2^{b(p)}} > q^{2^{b(q)-1}} = \ell. \tag{1}$$

To see this, note first that ℓ divides $f(2^k)$. With the first part of Solution 1 one can see that the integer $n = f(2^k)m/\ell$ also satisfies $d(n) = 2^k$. By the definition of $f(2^k)$ this implies that $n \geq f(2^k)$ so $m \geq \ell$. Since $p \neq q$ the inequality (1) follows.

Let the prime factorization of $f(2^{k+1})$ be given by $f(2^{k+1}) = \prod_p p^{r(p)}$ with $r(p) = 2^{s(p)} - 1$. Since we have $\sum_p s(p) = k + 1 > k = \sum_p b(p)$ there is a prime p with $s(p) > b(p)$. For any prime $q \neq p$ with $b(q) > 0$ we apply inequality (1) twice and get

$$q^{2^{s(q)}} > p^{2^{s(p)-1}} \geq p^{2^{b(p)}} > q^{2^{b(q)-1}},$$

which implies $s(q) \geq b(q)$. It follows that $s(q) \geq b(q)$ for all primes q, so $f(2^k) \mid f(2^{k+1})$.

N2

Consider a polynomial $P(x) = (x + d_1)(x + d_2) \cdot \ldots \cdot (x + d_9)$, where $d_1, d_2, \ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

Solution 1. Note that the statement of the problem is invariant under translations of x; hence without loss of generality we may suppose that the numbers $d_1, d_2, \ldots, d_9$ are positive.

The key observation is that there are only eight primes below 20, while $P(x)$ involves more than eight factors.

We shall prove that $N = d^8$ satisfies the desired property, where $d = \max\{d_1, d_2, \ldots, d_9\}$. Suppose for the sake of contradiction that there is some integer $x \geq N$ such that $P(x)$ is composed of primes below 20 only. Then for every index $i \in \{1, 2, \ldots, 9\}$ the number $x + d_i$ can be expressed as product of powers of the first 8 primes.

Since $x + d_i > x \geq d^8$ there is some prime power $f_i > d$ that divides $x + d_i$. Invoking the pigeonhole principle we see that there are two distinct indices i and j such that f_i and f_j are powers of the same prime number. For reasons of symmetry, we may suppose that $f_i \leq f_j$. Now both of the numbers $x + d_i$ and $x + d_j$ are divisible by f_i and hence so is their difference $d_i - d_j$. But as

$$0 < |d_i - d_j| \leq \max(d_i, d_j) \leq d < f_i,$$

this is impossible. Thereby the problem is solved.

Solution 2. Observe that for each index $i \in \{1, 2, \ldots, 9\}$ the product

$$D_i = \prod_{1 \leq j \leq 9, j \neq i} |d_i - d_j|$$

is positive. We claim that $N = \max\{D_1 - d_1, D_2 - d_2, \ldots, D_9 - d_9\} + 1$ satisfies the statement of the problem. Suppose there exists an integer $x \geq N$ such that all primes dividing $P(x)$ are smaller than 20. For each index i we reduce the fraction $(x + d_i)/D_i$ to lowest terms. Since $x + d_i > D_i$ the numerator of the fraction we thereby get cannot be 1, and hence it has to be divisible by some prime number $p_i < 20$.

By the pigeonhole principle, there are a prime number p and two distinct indices i and j such that $p_i = p_j = p$. Let p^{α_i} and p^{α_j} be the greatest powers of p dividing $x + d_i$ and $x + d_j$, respectively. Due to symmetry we may suppose $\alpha_i \leq \alpha_j$. But now p^{α_i} divides $d_i - d_j$ and hence also D_i, which means that all occurrences of p in the numerator of the fraction $(x + d_i)/D_i$ cancel out, contrary to the choice of $p = p_i$. This contradiction proves our claim.

Solution 3. Given a nonzero integer N as well as a prime number p we write $v_p(N)$ for the exponent with which p occurs in the prime factorization of $|N|$.

Evidently, if the statement of the problem were not true, then there would exist an infinite sequence (x_n) of positive integers tending to infinity such that for each $n \in \mathbb{Z}_+$ the integer $P(x_n)$ is not divisible by any prime number > 20. Observe that the numbers $-d_1, -d_2, \ldots, -d_9$ do not appear in this sequence.

Now clearly there exists a prime $p_1 < 20$ for which the sequence $v_{p_1}(x_n + d_1)$ is not bounded; thinning out the sequence (x_n) if necessary we may even suppose that

$$v_{p_1}(x_n + d_1) \longrightarrow \infty.$$

Repeating this argument eight more times we may similarly choose primes $p_2, \ldots, p_9 < 20$ and suppose that our sequence (x_n) has been thinned out to such an extent that $v_{p_i}(x_n + d_i) \longrightarrow \infty$ holds for $i = 2, \ldots, 9$ as well. In view of the pigeonhole principle, there are distinct indices i and j as well as a prime $p < 20$ such that $p_i = p_j = p$. Setting $k = v_p(d_i - d_j)$ there now has to be some n for which both $v_p(x_n + d_i)$ and $v_p(x_n + d_j)$ are greater than k. But now the numbers $x_n + d_i$ and $x_n + d_j$ are divisible by p^{k+1} whilst their difference $d_i - d_j$ is not – a contradiction.

Comment. This problem is supposed to be a relatively easy one, so one might consider adding the hypothesis that the numbers $d_1, d_2, \ldots, d_9$ be positive. Then certain merely technical issues are not going to arise while the main ideas required to solve the problems remain the same.

N3

Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

Answer. All functions f of the form $f(x) = \varepsilon x^d + c$, where ε is in $\{1, -1\}$, the integer d is a positive divisor of n, and c is an integer.

Solution. Obviously, all functions in the answer satisfy the condition of the problem. We will show that there are no other functions satisfying that condition.

Let f be a function satisfying the given condition. For each integer n, the function g defined by $g(x) = f(x) + n$ also satisfies the same condition. Therefore, by subtracting $f(0)$ from $f(x)$ we may assume that $f(0) = 0$.

For any prime p, the condition on f with $(x, y) = (p, 0)$ states that $f(p)$ divides p^n. Since the set of primes is infinite, there exist integers d and ε with $0 \leq d \leq n$ and $\varepsilon \in \{1, -1\}$ such that for infinitely many primes p we have $f(p) = \varepsilon p^d$. Denote the set of these primes by P. Since a function g satisfies the given condition if and only if $-g$ satisfies the same condition, we may suppose $\varepsilon = 1$.

The case $d = 0$ is easily ruled out, because 0 does not divide any nonzero integer. Suppose $d \geq 1$ and write n as $md + r$, where m and r are integers such that $m \geq 1$ and $0 \leq r \leq d - 1$. Let x be an arbitrary integer. For each prime p in P, the difference $f(p) - f(x)$ divides $p^n - x^n$. Using the equality $f(p) = p^d$, we get

$$p^n - x^n = p^r (p^d)^m - x^n \equiv p^r f(x)^m - x^n \equiv 0 \pmod{p^d - f(x)}$$

Since we have $r < d$, for large enough primes $p \in P$ we obtain

$$|p^r f(x)^m - x^n| < p^d - f(x).$$

Hence $p^r f(x)^m - x^n$ has to be zero. This implies $r = 0$ and $x^n = (x^d)^m = f(x)^m$. Since m is odd, we obtain $f(x) = x^d$.

Comment. If n is an even positive integer, then the functions f of the form

$$f(x) = \begin{cases} x^d + c \text{ for some integers,} \\ -x^d + c \text{ for the rest of integers,} \end{cases}$$

where d is a positive divisor of $n/2$ and c is an integer, also satisfy the condition of the problem. Together with the functions in the answer, they are all functions that satisfy the condition when n is even.

N4

For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n such that all the differences

$$t(n+a)-t(n), \quad t(n+a+1)-t(n+1), \quad \ldots, \quad t(n+2a-1)-t(n+a-1)$$

are divisible by 4.

Answer. $a = 1$, 3, or 5.

Solution. A pair (a, n) satisfying the condition of the problem will be called a *winning pair*. It is straightforward to check that the pairs $(1,1)$, $(3,1)$, and $(5,4)$ are winning pairs.

Now suppose that a is a positive integer not equal to 1, 3, and 5. We will show that there are no winning pairs (a, n) by distinguishing three cases.

Case 1: a is even. In this case we have $a = 2^\alpha d$ for some positive integer α and some odd d. Since $a \geq 2^\alpha$, for each positive integer n there exists an $i \in \{0, 1, \ldots, a-1\}$ such that $n + i = 2^{\alpha-1}e$, where e is some odd integer. Then we have $t(n+i) = t(2^{\alpha-1}e) = e$ and

$$t(n+a+i) = t(2^\alpha d + 2^{\alpha-1}e) = 2d + e \equiv e + 2 \pmod 4.$$

So we get $t(n+i) - t(n+a+i) \equiv 2 \pmod 4$, and (a, n) is not a winning pair.

Case 2: a is odd and $a > 8$. For each positive integer n, there exists an $i \in \{0, 1, \ldots, a-5\}$ such that $n + i = 2d$ for some odd d. We get

$$t(n+i) = d \not\equiv d + 2 = t(n+i+4) \pmod 4$$

and

$$t(n+a+i) = n+a+i \equiv n+a+i+4 = t(n+a+i+4) \pmod 4.$$

Therefore, the integers $t(n+a+i) - t(n+i)$ and $t(n+a+i+4) - t(n+i+4)$ cannot be both divisible by 4, and therefore there are no winning pairs in this case.

Case 3: $a = 7$. For each positive integer n, there exists an $i \in \{0, 1, \ldots, 6\}$ such that $n + i$ is either of the form $8k + 3$ or of the form $8k + 6$, where k is a nonnegative integer. But we have

$$t(8k+3) \equiv 3 \not\equiv 1 \equiv 4k + 5 = t(8k+3+7) \pmod 4$$

and

$$t(8k+6) = 4k+3 \equiv 3 \not\equiv 1 \equiv t(8k+6+7) \pmod 4.$$

Hence, there are no winning pairs of the form $(7, n)$.

N5

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m)-f(n)$ is divisible by $f(m-n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

Solution 1. Suppose that x and y are two integers with $f(x)<f(y)$. We will show that $f(x) \mid f(y)$. By taking $m=x$ and $n=y$ we see that

$$f(x-y) \mid |f(x)-f(y)| = f(y)-f(x) > 0,$$

so $f(x-y) \leq f(y)-f(x) < f(y)$. Hence the number $d=f(x)-f(x-y)$ satisfies

$$-f(y) < -f(x-y) < d < f(x) < f(y).$$

Taking $m=x$ and $n=x-y$ we see that $f(y) \mid d$, so we deduce $d=0$, or in other words $f(x)=f(x-y)$. Taking $m=x$ and $n=y$ we see that $f(x)=f(x-y) \mid f(x)-f(y)$, which implies $f(x) \mid f(y)$.

Solution 2. We split the solution into a sequence of claims; in each claim, the letters m and n denote arbitrary integers.

Claim 1. $f(n) \mid f(mn)$.

Proof. Since trivially $f(n) \mid f(1 \cdot n)$ and $f(n) \mid f((k+1)n)-f(kn)$ for all integers k, this is easily seen by using induction on m in both directions. □

Claim 2. $f(n) \mid f(0)$ and $f(n)=f(-n)$.

Proof. The first part follows by plugging $m=0$ into Claim 1. Using Claim 1 twice with $m=-1$, we get $f(n) \mid f(-n) \mid f(n)$, from which the second part follows. □

From Claim 1, we get $f(1) \mid f(n)$ for all integers n, so $f(1)$ is the minimal value attained by f. Next, from Claim 2, the function f can attain only a finite number of values since all these values divide $f(0)$.

Now we prove the statement of the problem by induction on the number N_f of values attained by f. In the base case $N_f \leq 2$, we either have $f(0) \neq f(1)$, in which case these two numbers are the only values attained by f and the statement is clear, or we have $f(0)=f(1)$, in which case we have $f(1) \mid f(n) \mid f(0)$ for all integers n, so f is constant and the statement is obvious again.

For the induction step, assume that $N_f \geq 3$, and let a be the least positive integer with $f(a)>f(1)$. Note that such a number exists due to the symmetry of f obtained in Claim 2.

Claim 3. $f(n) \neq f(1)$ if and only if $a \mid n$.

Proof. Since $f(1) = \cdots = f(a-1) < f(a)$, the claim follows from the fact that

$$f(n) = f(1) \iff f(n+a) = f(1).$$

So it suffices to prove this fact.

Assume that $f(n) = f(1)$. Then $f(n+a) \mid f(a) - f(-n) = f(a) - f(n) > 0$, so $f(n+a) \leq f(a) - f(n) < f(a)$; in particular the difference $f(n+a) - f(n)$ is stricly smaller than $f(a)$. Furthermore, this difference is divisible by $f(a)$ and nonnegative since $f(n) = f(1)$ is the least value attained by f. So we have $f(n+a) - f(n) = 0$, as desired. For the converse direction we only need to remark that $f(n+a) = f(1)$ entails $f(-n-a) = f(1)$, and hence $f(n) = f(-n) = f(1)$ by the forward implication. □

We return to the induction step. So let us take two arbitrary integers m and n with $f(m) \leq f(n)$. If $a \nmid m$, then we have $f(m) = f(1) \mid f(n)$. On the other hand, suppose that $a \mid m$; then by Claim 3 $a \mid n$ as well. Now define the function $g(x) = f(ax)$. Clearly, g satisfies the conditions of the problem, but $N_g < N_f - 1$, since g does not attain $f(1)$. Hence, by the induction hypothesis, $f(m) = g(m/a) \mid g(n/a) = f(n)$, as desired.

Comment. After the fact that f attains a finite number of values has been established, there are several ways of finishing the solution. For instance, let $f(0) = b_1 > b_2 > \cdots > b_k$ be all these values. One may show (essentially in the same way as in Claim 3) that the set $S_i = \{n : f(n) \geq b_i\}$ consists exactly of all numbers divisible by some integer $a_i \geq 0$. One obviously has $a_i \mid a_{i-1}$, which implies $f(a_i) \mid f(a_{i-1})$ by Claim 1. So, $b_k \mid b_{k-1} \mid \cdots \mid b_1$, thus proving the problem statement.

Moreover, now it is easy to describe all functions satisfying the conditions of the problem. Namely, all these functions can be constructed as follows. Consider a sequence of nonnegative integers $a_1, a_2, \ldots, a_k$ and another sequence of positive integers $b_1, b_2, \ldots, b_k$ such that $|a_k| = 1$, $a_i \neq a_j$ and $b_i \neq b_j$ for all $1 \leq i < j \leq k$, and $a_i \mid a_{i-1}$ and $b_i \mid b_{i-1}$ for all $i = 2, \ldots, k$. Then one may introduce the function

$$f(n) = b_{i(n)}, \qquad \text{where} \quad i(n) = \min\{i : a_i \mid n\}.$$

These are all the functions which satisfy the conditions of the problem.

N6

Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)}-1$ divides $3^{P(n)}-1$. Prove that $Q(x)$ is a constant polynomial.

Solution. First we show that there exists an integer d such that for all positive integers n we have $\gcd\big(P(n), Q(n)\big) \leq d$.

Since $P(x)$ and $Q(x)$ are coprime (over the polynomials with rational coefficients), Euclid's algorithm provides some polynomials $R_0(x), S_0(x)$ with rational coefficients such that $P(x)R_0(x)-Q(x)S_0(x)=1$. Multiplying by a suitable positive integer d, we obtain polynomials $R(x)=d\cdot R_0(x)$ and $S(x)=d\cdot S_0(x)$ with integer coefficients for which $P(x)R(x)-Q(x)S(x)=d$. Then we have $\gcd\big(P(n), Q(n)\big) \leq d$ for any integer n.

To prove the problem statement, suppose that $Q(x)$ is not constant. Then the sequence $Q(n)$ is not bounded and we can choose a positive integer m for which

$$M = 2^{Q(m)} - 1 \geq 3^{\max\{P(1), P(2), \ldots, P(d)\}}. \tag{1}$$

Since $M = 2^{Q(n)}-1 \mid 3^{P(n)}-1$, we have $2, 3 \nmid M$. Let a and b be the multiplicative orders of 2 and 3 modulo M, respectively. Obviously, $a = Q(m)$ since the lower powers of 2 do not reach M. Since M divides $3^{P(m)}-1$, we have $b \mid P(m)$. Then $\gcd(a,b) \leq \gcd\big(P(m), Q(m)\big) \leq d$. Since the expression $ax-by$ attains all integer values divisible by $\gcd(a,b)$ when x and y run over all nonnegative integer values, there exist some nonnegative integers x, y such that $1 \leq m+ax-by \leq d$.

By $Q(m+ax) \equiv Q(m) \pmod a$ we have

$$2^{Q(m+ax)} \equiv 2^{Q(m)} \equiv 1 \pmod M$$

and therefore

$$M \mid 2^{Q(m+ax)}-1 \mid 3^{P(m+ax)}-1.$$

Then, by $P(m+ax-by) \equiv P(m+ax) \pmod b$ we have

$$3^{P(m+ax-by)} \equiv 3^{P(m+ax)} \equiv 1 \pmod M.$$

Since $P(m+ax-by) > 0$ this implies $M \leq 3^{P(m+ax-by)}-1$. But $P(m+ax-by)$ is listed among $P(1), P(2), \ldots, P(d)$, so

$$M < 3^{P(m+ax-by)} \leq 3^{\max\{P(1), P(2), \ldots, P(d)\}}$$

which contradicts (1).

Comment. We present another variant of the solution above.

Denote the degree of P by k and its leading coefficient by p. Consider any positive integer n and let $a = Q(n)$. Again, denote by b the multiplicative order of 3 modulo $2^a - 1$. Since $2^a - 1 \mid 3^{P(n)} - 1$, we have $b \mid P(n)$. Moreover, since $2^{Q(n+at)} - 1 \mid 3^{P(n+at)} - 1$ and $a = Q(n) \mid Q(n + at)$ for each positive integer t, we have $2^a - 1 \mid 3^{P(n+at)} - 1$, hence $b \mid P(n + at)$ as well.

Therefore, b divides $\gcd\{P(n + at) : t \geq 0\}$; hence it also divides the number

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} P(n + ai) = p \cdot k! \cdot a^k.$$

Finally, we get $b \,\big|\, \gcd\big(P(n), k! \cdot p \cdot Q(n)^k\big)$, which is bounded by the same arguments as in the beginning of the solution. So $3^b - 1$ is bounded, and hence $2^{Q(n)} - 1$ is bounded as well.

N7

Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.

Solution 1. For rational numbers p_1/q_1 and p_2/q_2 with the denominators q_1, q_2 not divisible by p, we write $p_1/q_1 \equiv p_2/q_2 \pmod{p}$ if the numerator $p_1q_2 - p_2q_1$ of their difference is divisible by p.

We start with finding an explicit formula for the residue of S_a modulo p. Note first that for every $k = 1, \ldots, p-1$ the number $\binom{p}{k}$ is divisible by p, and

$$\frac{1}{p}\binom{p}{k} = \frac{(p-1)(p-2)\cdots(p-k+1)}{k!} \equiv \frac{(-1)\cdot(-2)\cdots(-k+1)}{k!} = \frac{(-1)^{k-1}}{k} \pmod{p}$$

Therefore, we have

$$S_a = -\sum_{k=1}^{p-1} \frac{(-a)^k(-1)^{k-1}}{k} \equiv -\sum_{k=1}^{p-1}(-a)^k \cdot \frac{1}{p}\binom{p}{k} \pmod{p}.$$

The number on the right-hand side is integer. Using the binomial formula we express it as

$$-\sum_{k=1}^{p-1}(-a)^k \cdot \frac{1}{p}\binom{p}{k} = -\frac{1}{p}\left(-1-(-a)^p + \sum_{k=0}^{p}(-a)^k\binom{p}{k}\right) = \frac{(a-1)^p - a^p + 1}{p}$$

since p is odd. So, we have

$$S_a \equiv \frac{(a-1)^p - a^p + 1}{p} \pmod{p}.$$

Finally, using the obtained formula we get

$$\begin{aligned} S_3 + S_4 - 3S_2 &\equiv \frac{(2^p - 3^p + 1) + (3^p - 4^p + 1) - 3(1^p - 2^p + 1)}{p} \\ &= \frac{4\cdot 2^p - 4^p - 4}{p} = -\frac{(2^p-2)^2}{p} \pmod{p}. \end{aligned}$$

By FERMAT's theorem, $p \mid 2^p - 2$, so $p^2 \mid (2^p-2)^2$ and hence $S_3 + S_4 - 3S_2 \equiv 0 \pmod{p}$.

Solution 2. One may solve the problem without finding an explicit formula for S_a. It is enough to find the following property.

Lemma. For every integer a, we have $S_{a+1} \equiv S_{-a} \pmod{p}$.

Proof. We expand S_{a+1} using the binomial formula as

$$S_{a+1} = \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} a^j = \sum_{k=1}^{p-1} \left(\frac{1}{k} + \sum_{j=1}^{k} a^j \cdot \frac{1}{k} \binom{k}{j} \right) = \sum_{k=1}^{p-1} \frac{1}{k} + \sum_{j=1}^{p-1} a^j \sum_{k=j}^{p-1} \frac{1}{k} \binom{k}{j} a^k.$$

Note that $\frac{1}{k} + \frac{1}{p-k} = \frac{p}{k(p-k)} \equiv 0 \pmod{p}$ for all $1 \le k \le p-1$; hence the first sum vanishes modulo p. For the second sum, we use the relation $\frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{k-1}{j-1}$ to obtain

$$S_{a+1} \equiv \sum_{j=1}^{p-1} \frac{a^j}{j} \sum_{k=1}^{p-1} \binom{k-1}{j-1} \pmod{p}.$$

Finally, from the relation

$$\sum_{k=1}^{p-1} \binom{k-1}{j-1} = \binom{p-1}{j} = \frac{(p-1)(p-2)\dots(p-j)}{j!} \equiv (-1)^j \pmod{p}$$

we obtain

$$S_{a+1} \equiv \sum_{j=1}^{p-1} \frac{a^j(-1)^j}{j!} = S_{-a}. \qquad \square$$

Now we turn to the problem. Using the lemma we get

$$S_3 - 3S_2 \equiv S_{-2} - 3S_2 = \sum_{\substack{1 \le k \le p-1 \\ k \text{ is even}}} \frac{-2 \cdot 2^k}{k} + \sum_{\substack{1 \le k \le p-1 \\ k \text{ is odd}}} \frac{-4 \cdot 2^k}{k} \pmod{p}. \tag{1}$$

The first sum in (1) expands as

$$\sum_{\ell=1}^{(p-1)/2} \frac{-2 \cdot 2^{2\ell}}{2\ell} = -\sum_{\ell=1}^{(p-1)/2} \frac{4^\ell}{\ell}.$$

Next, using Fermat's theorem, we expand the second sum in (1) as

$$-\sum_{\ell=1}^{(p-1)/2} \frac{2^{2\ell+1}}{2\ell-1} \equiv -\sum_{\ell=1}^{(p-1)/2} \frac{2^{p+2\ell}}{p+2\ell-1} = -\sum_{m=(p+1)/2}^{p-1} \frac{2 \cdot 4^m}{2m} = -\sum_{m=(p+1)/2}^{p-1} \frac{4^m}{m} \pmod{p}$$

(here we set $m = \ell + \frac{p-1}{2}$). Hence,

$$S_3 - 3S_2 \equiv -\sum_{\ell=1}^{(p-1)/2} \frac{4^\ell}{\ell} - \sum_{m=(p+1)/2}^{p-1} \frac{4^m}{m} = -S_4 \pmod{p}.$$

N8

Let k be a positive integer and set $n = 2^k + 1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1, \ldots, a_{n-1}$ of the numbers $1, 2, \ldots, n-1$ and a sequence of integers $g_1, g_2, \ldots, g_{n-1}$ such that n divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$, where we set $a_n = a_1$.

Solution. Let $N = \{1, 2, \ldots, n-1\}$. For $a, b \in N$, we say that b *follows* a if there exists an integer g such that $b \equiv g^a \pmod{n}$ and denote this property as $a \to b$. This way we have a directed graph with N as set of vertices. If $a_1, \ldots, a_{n-1}$ is a permutation of $1, 2, \ldots, n-1$ such that $a_1 \to a_2 \to \ldots \to a_{n-1} \to a_1$ then this is a *Hamiltonian cycle* in the graph.

Step I. First consider the case when n is composite. Let $n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ be its prime factorization. All primes p_i are odd.

Suppose that $\alpha_i > 1$ for some i. For all integers a, g with $a \geq 2$, we have $g^a \not\equiv p_i \pmod{p_i^2}$, because g^a is either divisible by p_i^2 or it is not divisible by p_i. It follows that in any Hamiltonian cycle p_i comes immediately after 1. The same argument shows that $2p_i$ also should come immediately after 1, which is impossible. Hence, there is no Hamiltonian cycle in the graph.

Now suppose that n is square-free. We have $n = p_1 p_2 \ldots p_s > 9$ and $s \geq 2$. Assume that there exists a Hamiltonian cycle. There are $\frac{n-1}{2}$ even numbers in this cycle, and each number which follows one of them should be a quadratic residue modulo n. So, there should be at least $\frac{n-1}{2}$ nonzero quadratic residues modulo n. On the other hand, for each p_i there exist exactly $\frac{p_i+1}{2}$ quadratic residues modulo p_i; by the Chinese Remainder Theorem, the number of quadratic residues modulo n is exactly $\frac{p_1+1}{2} \cdot \frac{p_2+1}{2} \cdot \ldots \cdot \frac{p_s+1}{2}$, including 0. Then we have a contradiction by

$$\frac{p_1+1}{2} \cdot \frac{p_2+1}{2} \cdot \ldots \cdot \frac{p_s+1}{2} \leq \frac{2p_1}{3} \cdot \frac{2p_2}{3} \cdot \ldots \cdot \frac{2p_s}{3} = \left(\frac{2}{3}\right)^s n \leq \frac{4n}{9} < \frac{n-1}{2}.$$

This proves the "if"-part of the problem.

Step II. Now suppose that n is prime. For any $a \in N$, denote by $\nu_2(a)$ the exponent of 2 in the prime factorization of a, and let $\mu(a) = \max\{t \in [0, k] \mid 2^t \to a\}$.

Lemma. For any $a, b \in N$, we have $a \to b$ if and only if $\nu_2(a) \leq \mu(b)$.

Proof. Let $\ell = \nu_2(a)$ and $m = \mu(b)$.

Suppose $\ell \leq m$. Since b follows 2^m, there exists some g_0 such that $b \equiv g_0^{2^m} \pmod{n}$. By $\gcd(a, n-1) = 2^\ell$ there exist some integers p and q such that $pa - q(n-1) = 2^\ell$. Choosing $g = g_0^{2^{m-\ell}p}$ we have $g^a = g_0^{2^{m-\ell}pa} = g_0^{2^m + 2^{m-\ell}q(n-1)} \equiv g_0^{2^m} \equiv b \pmod{n}$ by Fermat's theorem. Hence, $a \to b$.

To prove the reverse statement, suppose that $a \to b$, so $b \equiv g^a \pmod{n}$ with some g. Then $b \equiv (g^{a/2^\ell})^{2^\ell}$, and therefore $2^\ell \to b$. By the definition of $\mu(b)$, we have $\mu(b) \geq \ell$. The lemma is proved. $\square$

Now for every i with $0 \le i \le k$, let

$$\begin{aligned} A_i &= \{a \in N \mid \nu_2(a) = i\}, \\ B_i &= \{a \in N \mid \mu(a) = i\}, \\ \text{and } C_i &= \{a \in N \mid \mu(a) \ge i\} = B_i \cup B_{i+1} \cup \ldots \cup B_k. \end{aligned}$$

We claim that $|A_i| = |B_i|$ for all $0 \le i \le k$. Obviously we have $|A_i| = 2^{k-i-1}$ for all $i = 0, \ldots, k-1$, and $|A_k| = 1$. Now we determine $|C_i|$. We have $|C_0| = n-1$ and by Fermat's theorem we also have $C_k = \{1\}$, so $|C_k| = 1$. Next, notice that $C_{i+1} = \{x^2 \mod n \mid x \in C_i\}$. For every $a \in N$, the relation $x^2 \equiv a \pmod n$ has at most two solutions in N. Therefore we have $2|C_{i+1}| \le |C_i|$, with the equality achieved only if for every $y \in C_{i+1}$, there exist distinct elements $x, x' \in C_i$ such that $x^2 \equiv x'^2 \equiv y \pmod n$ (this implies $x + x' = n$). Now, since $2^k|C_k| = |C_0|$, we obtain that this equality should be achieved in each step. Hence $|C_i| = 2^{k-i}$ for $0 \le i \le k$, and therefore $|B_i| = 2^{k-i-1}$ for $0 \le i \le k-1$ and $|B_k| = 1$.

From the previous arguments we can see that for each $z \in C_i$ $(0 \le i < k)$ the equation $x^2 \equiv z^2 \pmod n$ has two solutions in C_i, so we have $n - z \in C_i$. Hence, for each $i = 0, 1, \ldots, k-1$, exactly half of the elements of C_i are odd. The same statement is valid for $B_i = C_i \setminus C_{i+1}$ for $0 \le i \le k-2$. In particular, each such B_i contains an odd number. Note that $B_k = \{1\}$ also contains an odd number, and $B_{k-1} = \{2^k\}$ since C_{k-1} consists of the two square roots of 1 modulo n.

Step III. Now we construct a Hamiltonian cycle in the graph. First, for each i with $0 \le i \le k$, connect the elements of A_i to the elements of B_i by means of an arbitrary bijection. After performing this for every i, we obtain a subgraph with all vertices having in-degree 1 and out-degree 1, so the subgraph is a disjoint union of cycles. If there is a unique cycle, we are done. Otherwise, we modify the subgraph in such a way that the previous property is preserved and the number of cycles decreases; after a finite number of steps we arrive at a single cycle.

For every cycle C, let $\lambda(C) = \min_{c \in C} \nu_2(c)$. Consider a cycle C for which $\lambda(C)$ is maximal. If $\lambda(C) = 0$, then for any other cycle C' we have $\lambda(C') = 0$. Take two arbitrary vertices $a \in C$ and $a' \in C'$ such that $\nu_2(a) = \nu_2(a') = 0$; let their direct successors be b and b', respectively. Then we can unify C and C' to a single cycle by replacing the edges $a \to b$ and $a' \to b'$ by $a \to b'$ and $a' \to b$.

Now suppose that $\lambda = \lambda(C) \ge 1$; let $a \in C \cap A_\lambda$. If there exists some $a' \in A_\lambda \setminus C$, then a' lies in another cycle C' and we can merge the two cycles in exactly the same way as above. So, the only remaining case is $A_\lambda \subset C$. Since the edges from A_λ lead to B_λ, we get also $B_\lambda \subset C$. If $\lambda \ne k-1$ then B_λ contains an odd number; this contradicts the assumption $\lambda(C) > 0$. Finally, if $\lambda = k-1$, then C contains 2^{k-1} which is the only element of A_{k-1}. Since $B_{k-1} = \{2^k\} = A_k$ and $B_k = \{1\}$, the cycle C contains the path $2^{k-1} \to 2^k \to 1$ and it contains an odd number again. This completes the proof of the "only if"-part of the problem.

Comment 1. The lemma and the fact $|A_i| = |B_i|$ together show that for every edge $a \to b$ of the Hamiltonian cycle, $\nu_2(a) = \mu(b)$ must hold. After this observation, the Hamiltonian cycle can be built in many ways. For instance, it is possible to select edges from A_i to B_i for $i = k, k-1, \ldots, 1$ in such a way that they form disjoint paths; at the end all these paths will have odd endpoints. In the final step, the paths can be closed to form a unique cycle.

Comment 2. Step II is an easy consequence of some basic facts about the multiplicative group modulo the prime $n = 2^k + 1$. The Lemma follows by noting that this group has order 2^k, so the a-th powers are exactly the $2^{\nu_2(a)}$-th powers. Using the existence of a primitive root g modulo n one sees that the map from $\{1, 2, \ldots, n-1\}$ to itself that sends a to $g^a \bmod n$ is a bijection that sends A_i to B_i for each $i \in \{0, \ldots, k\}$.